Optimal Consumption and Investment Strategies under Wealth Ratcheting

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ABSTRACT

Individuals driven by capital accumulation may be reluctant to experience large wealth downfalls. Implications for optimal consumption and investment policies are explored in a dynamic setting where wealth is restrained from falling below a fraction of its all-time high. Risky investment regulates wealth growth and mitigates the ratchet effect of the constraint, and may decrease as wealth approaches its maximum. The correspondence found between habit formation over consumption and wealth ratcheting provides a rational explanation for the extensive use of such a practice in investment management. An extension embeds the spirit of capitalism using wealth as an index for social status.

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1 INTRODUCTION

The desire for wealth accumulation is well established in the literature. According to Max Weber [36], "man is dominated by the making of money, by acquisition as the ultimate purpose of his life. Economic acquisition is no longer subordinated to man as the means for the satisfaction of his material needs". The recent explosion and success of capital guarantee funds suggest that investors are looking for downside protection but at the same time upside potential, in particular for long term investments such as 401(k)'s and especially during financial troubled times. Fund trusts and institutions such as a university or a foundation may also seek asset preservation. When a non-profit organization receives an endowment and other long-term funding, it has to manage these resources prudently by establishing a spending policy that accommodates the need for asset protection and portfolio growth. Usually donors require endowment assets to be kept permanently and prohibit grantees from using or borrowing against principals. The aim of such spending rules is to preserve financial independence and to avoid the purchasing power erosion over time[1]. Investors often use all-time record levels as a benchmarks to measure fund performances and high-water marks[2] are common in the investment management industry. This has prompted some financial services firms to offer their customers the following portfolio insurance strategy: an investor who stays invested until the fund matures is guaranteed to receive a value equal to the highest value of the fund ever achieved, even if the fund's daily value has fallen since its highest point.

In this paper, we analyze the intertemporal investment-consumption rules for an infinite lived individual maximizing her expected discounted utility under wealth ratcheting. Namely, the agent does not tolerate losing more than a fixed percentage of her all-time high level of wealth[3]. Our motivation for such specification of preferences is twofold. The first one deals with the investor's ego that dictates an extreme form of loss aversion over wealth since utility can be defined to be minus infinity if the drawdown constraint is violated. The concept of loss aversion postulates that the impact of a loss is greater than that of an equally sized gain. Investors may like to exhibit their superior forecasts about the market movements, but are less keen on acknowledging some wrong predictions. After a significant loss in the stock market, an investor may lament or blame herself about her decision to invest in stocks. At a personal level, she may interpret her loss as a sign that after all, she is not the first rate investor she thought she was, thus inflicting a painful blow to her ego. At a social level, she may feel humiliation as the news spread among colleagues, friends and family members. The second motivation captures the fact that when an investor's wealth goes up, she may psychologically commit part of the profits made in order to finance future expenditures, including sometimes important lifetime decisions such as early retirement. Being forced to revise downward her plans after a sudden drop in wealth may turn out to be quite painful.

The key intuition behind most of the results is driven by two effects. First, as in any portfolio selection problem under market restrictions, the agent is concerned with hedging motives that in the future the constraint may be binding. As a benchmark, hedging concerns are addressed in a simpler framework when the investor is required to maintain her wealth above a *fixed* floor (university

[1] In the US, trustees and charity professionals who run foundations are only obliged to spend as little as 5% a year of the capital. In many foundations, capricious and poorly thought out projects or programs were undertaken to fulfill the interests of trustee managers not the wishes of the founder [17].

[2] The high-water mark is a target value that can depend on the current asset value of the fund. It is periodically adjusted due to withdraws, allocated expenses and a contractual growth rate. In the simplest case, the high-water mark is the highest level the asset has reached in the past.

[3] This constraint was first introduced by Grossman and Zhou [21] to examine the problem of maximizing the long term growth rate of expected utility of final wealth. Their analysis is quite insightful but they do not allow for endogenous withdraws from the fund to finance intermediate consumption. Cvitanic and Karatzas [12] extend their work to a more general class of stochastic processes.

charter requirement). Essentially, risk aversion is enhanced, which leads to smaller stock holdings and lower consumption plans with respect to the unconstrained case. Both optimal allocations are found increasing in wealth. Second, the drawdown constraint displays a ratcheting feature since each time financial wealth reaches a new record high, the minimum floor rises and the restriction becomes more stringent. The agent has two margins of adjustment at her disposal to regulate the growth of her wealth: consumption and risky investment. The latter is the most sensitive of the two as it governs the diffusion component of the wealth process. The optimal solution of the model reflects the trade-off between consuming today and deferring consumption to take advantage of investing in the stock market, which may be thwarted by the presence of the ratchet. We derive conditions under which, as wealth approaches its all-time high, the fraction of wealth invested in stocks decreases and possibly approaches zero. This last case occurs when the investor is fairly impatient, namely when the pure time discount rate exceeds the interest rate, and the drawdown coefficient is high. Impatient agents wish to consume a large fraction of their current wealth so they are very reluctant to ratchet up the minimum floor; as a consequence, the optimal wealth process never achieves a new height.

Tracking wealth movements, the optimal consumption policy exhibits a ratcheting behavior and large drawdowns from its all-time consumption level are prohibited. We emphasize the correspondence between wealth ratcheting and habit formation in the spirit of Duesenberry [15]. This twin ratcheting is an important result that rationalizes the loss aversion over wealth for an investor who aims to maintain her standard of living. An extension of the basic model embeds the spirit of capitalism by including wealth, an index of social status, inside the utility function[4]. Persistent benefits derived from building up status lead to a more aggressive risky investment policy whereas consumption becomes less appealing.

This paper builds on the dynamic portfolio choice literature. Early works on optimal consumption-investment allocations in a frictionless market and no borrowing restrictions include Samuelson [30] and Merton [28]. Further, attention has been paid on more real world situations where investors face constraints in their portfolio investments[5]. In general, the optimal strategy differs from the unconstrained one as the agent aims at hedging against the constraint (at some cost) since even though the constraint may not be binding, there is a possibility that it does in the future. Recent papers focus on portfolio allocations under wealth performance relative to an exogenous benchmark such as in Browne [4] and Tepla [33] or subject to growth objectives required by the decision maker as in Hellwig [25]. In Carpenter [6], the fund manager is compensated with a call option on the wealth she manages with a benchmark index as strike price. The author shows that the option compensation does not necessarily lead to more risk seeking. Goetzmann, Ingersoll and Ross [19] study hedge fund compensation schemes when managers perceive a regular fee proportional to the portfolio asset value and an incentive fee based on the fund return each year in excess of the high-water mark. Consistent with empirical evidence, they obtain that a significant proportion of managers compensation can be attributed to the incentive fee, in particular for high volatility asset funds for which high manager skills are required.

The paper is also related to the trend of research that strives to provide some alternative to the usual time separable von Neumann-Morgenstein preferences whose performance has been poor from an empirical point of view. Such preferences have failed to explain important facts about stock returns;

[4]For instance see Bakshi and Chen [1] and Smith [31].

[5]Cvitanic and Karatzas [11] and Cuoco [9] develop a general martingale approach to cope with convex contemporaneous constraints on trading strategies which includes the case of incomplete markets and prohibited short sales. Cuoco and Liu [10] analyze the optimal consumption portfolio choice problem under margin requirements and evaluate the cost of the constraint. He and Pages [24] and El Karaoui and JeanBlanc-Picqué [18] treat the case of non-negative wealth in presence of labor income. Grossman and Villa [20] followed by Villa and Zariphopoulou [35] study the consumption-portfolio problem for a CRRA investor facing a leverage constraint.

for instance the equity premium (see Mehra and Prescott [27]). A popular attempt to provide a more sophisticated specification for utility is habit formation[6] that postulates that agents not only derive utility from current consumption but also from consumption history, typically captured by a standard of living index. For tractability reasons, many models assume that the agent derives utility from the excess between current consumption and the habit level. If the marginal utility at zero is infinite, the standard of living index acts as a floor level below which current consumption does not fall. This addictive feature - optimal consumption levels can only increase across time regardless of the state of the economy- is not supported by empirical evidence. Detemple and Karatzas [14] address this issue and investigate the case of finite marginal utility of consumption at zero when imposing a non-negativity constraint on consumption plans. When the shadow price of consumption is high, the agent optimally reduces her consumption along with her standard of living and the associated "cost" of habits as well. An alternative approach proposed by Dybvig [16] is to ratchet current consumption. Originally, Duesenberry [15] argued that consumption may not be entirely reversible over time but instead may increase along with income and decline less than proportionally with it. Dybvig [16] formalizes this idea by looking at an extreme form of habit formation where consumption is prevented from falling over time. It is straightforward to extend Dybvig's analysis to the case of an agent who is intolerant to any decline in consumption that exceeds a fixed proportion of her all-time consumption. We show that ratcheting wealth induces a ratcheting behavior on consumption with a strong parallel with Dybvig [16]. Essentially, the framework developed in this paper is a mirror image of the one proposed in Dybvig [16].

Finally, our model borrows concepts from the psychology literature. The idea that investors are more sensitive to wealth reductions than wealth increases (loss aversion) is the central feature of the prospect theory developed by Kahneman and Tversky ([27] and [34]). Barberis, Huang and Santos [2] explore the implications on asset prices of loss aversion by considering an investor who derives utility not only from consumption but also from changes in the value of her financial wealth. Here we take a different approach. The benchmark for loss aversion experienced by the investor is the all-high level of wealth.

The paper is organized as follows. Section 2 describes the economic setting and contains an heuristic derivation of the optimal consumption and portfolio allocations for a CRRA utility agent. In section 3, we discuss an extension of the basic model that embeds the spirit of capitalism using wealth as a proxy for social status. Section 4 displays a verification theorem that formally proves that the heuristic solution is indeed valid. Section 5 concludes. Proofs of all results are collected in the appendix.

2 THE ECONOMIC SETTING

Time is continuous. An infinitely lived investor, who is reluctant to let her wealth fall more than a fraction of its historical maximum, has to optimally allocate her wealth between a risk-free bond, a risky asset and consumption.

Individual Preferences. There is a single perishable good available for consumption, the numéraire. Preferences are represented by a time additive utility function

$$U(c) = E_0\left[\int_0^\infty u(c_s)e^{-\theta' s}ds\right],$$

[6]See for instance Sundaresan [32], Constantidines [8], Detemple and Zapatero [13], Campbell and Cochrane [5].

where the instantaneous utility function u is twice continuously differentiable, increasing and strictly concave and θ' denotes the subjective time discount rate. In addition, u satisfies the following Inada conditions: $\lim_{c \to 0^+} u'(c) = \infty$ and $\lim_{c \to \infty} u'(c) = 0$. In the sequel, we focus our analysis on an individual with constant relative risk aversion preferences

$$u(c) = \begin{cases} \frac{c^{1-b}}{1-b}, & b \neq 1 \\ \ln c, & b = 1. \end{cases}$$

Following Cuoco [9], in order to ensure that the expected utility U is well defined, we require consumption plans $\widehat{c}$ to satisfy[7]

$$\min \left\{ E_0 \left[\int_0^\infty u(\widehat{c}_s)^+ e^{-\theta s} ds \right], E_0 \left[\int_0^\infty u(\widehat{c}_s)^- e^{-\theta s} ds \right] \right\} < \infty, \tag{1}$$

where x^+ (respectively x^-) denotes the positive (respectively negative) part of x.

Financial Market. There are two securities available in the financial market:

- a risk-free bond whose price B evolves according to

$$dB_s = r' B_s ds,$$

where r' is the constant interest rate, and,

- an index modeled by a risky security whose price S follows a geometric Brownian motion

$$dS_s = S_s \left(\mu' ds + \sigma dw_s \right),$$

subject to some given initial stock price S_0, where dw_s is the increment of a standard Wiener process w, μ' is the mean return of the stock index S and σ^2 is its instantaneous variance. All the stochastic processes considered in the paper are assumed to be adapted on a common filtered probability space whose filtration is the one induced by the observations of w.

Let $\widehat{x}$ and $\widehat{z}$ be respectively the *amount* of dollars invested in the riskless bond B and risky security S, so that the wealth process $\widehat{W}$ is equal to $\widehat{x} + \widehat{z}$. A consumption plan $\widehat{c}$ is feasible if there is a trading strategy $\widehat{z}$ such that

$$\begin{aligned} d\widehat{W}_s &= (r'\widehat{W}_s - \widehat{c}_s + \widehat{z}_s(\mu' - r'))ds + \sigma \widehat{z}_s dw_s, \\ \widehat{W}_s &> -\underline{K}, \end{aligned} \tag{2}$$

with $\underline{K} > 0$. The condition $\widehat{W}_s > -\underline{K}$ rules out arbitrage opportunities, such as doubling strategies presented in Harrison and Kreps [22].

Drawdown Constraint. For $\lambda > 0$, define

$$\widehat{M}_t = \sup_{0 \le s \le t} \{ \widehat{M}_0 e^{\lambda t}, \widehat{W}_s e^{\lambda(t-s)}; 0 \le s \le t \}.$$

λ is the (minimum) growth rate of the floor required by the investor. As introduced in Grossman and Zhou [21], the drawdown constraint is

$$\widehat{W}_s \ge \alpha \widehat{M}_s, \tag{3}$$

[7] This assumption is only needed for the logarithmic preferences case. When $u(c)$ has a constant sign, Lebesgue Monotone Convergence Theorem can be applied directly to prove step 4 of the Verification Theorem provided in section 4 of the paper.

for some α in $(0,1]$. This constraint indicates that the investor is reluctant to let her wealth fall below a fraction of its maximum to date adjusted for some minimum growth rate λ. Grossman and Zhou [21] argue that a large drawdown (typically above 25 percent) is often a reason for firing fund managers. In the investment management industry, a realistic estimate of α ranges from 75 to 88 percent. In practice, different values of α may apply to different types of traders. For instance, for proprietary traders (internal hedge fund traders) who invest money belonging to their company, α can depend on the target amount of money a trader is required to generate during the year and could be as high as 94 percent. Stop loss strategies are often used by traders for downside protection. These strategies allow an investor to automatically unwind a position; they can be very effective at limiting losses if the market moves quickly against her.

We first review the main results for the unconstrained problem studied by Merton [28].

2.1 Merton Problem

Within our financial market framework, the Merton problem [28] for a CRRA investor is

$$F(\widehat{W}_t) = \max_{(\widehat{c},\widehat{z})} E_t \left[\int_t^\infty \frac{\widehat{c}_s^{1-b}}{1-b} e^{-\theta'(s-t)} ds \right],$$

subject to the budget constraint (2) and $\widehat{W}_t > 0$ given.

Merton [28] shows that the optimal strategy (c^f, z^f) is linear in wealth with $\frac{\widehat{z}_s^f}{\widehat{W}_s} = \frac{\mu'-r'}{b\sigma^2}$ and $\frac{\widehat{c}_s^f}{\widehat{W}_s} = \frac{1}{A}$, provided that $A^{-1} = \frac{\theta'}{b} + \frac{b-1}{b}\left(r' + \frac{(\mu'-r')^2}{2b\sigma^2}\right) > 0$. The optimal wealth process $\widehat{W}^f$ is a geometric Brownian motion whose dynamics are

$$d\widehat{W}_t^f = \widehat{W}_t^f \left((r' - \frac{1}{A} + \frac{(\mu'-r')^2}{b\sigma^2})dt + \frac{\mu'-r'}{b\sigma} dw_t \right).$$

In order to gain insights about the effects of the drawdown constraint (3), we examine a simpler consumption-portfolio choice problem where wealth is required to be kept above a *fixed* minimum floor adjusted for some required growth rate. This "natural" benchmark allows us to isolate and quantify hedging motives.

2.2 Benchmark Case: Fixed Minimum Floor Problem

Consider a foundation whose charter stipulates that the endowment $\alpha\underline{M} > 0$ adjusted for some required growth rate $\lambda > 0$ cannot be used for expenditures (only the returns are eligible). No other constraint is assumed regarding the growth objectives of the trust fund of a charitable foundation or a university. At any time t, wealth $\widehat{W}_t$ must be maintained above a minimum level $\alpha\underline{M}e^{\lambda t}$. Let us define $W_t \equiv \widehat{W}_t e^{-\lambda t}$, $c_t \equiv \widehat{c}_t e^{-\lambda t}$ and $z_t \equiv \widehat{z}_t e^{-\lambda t}$. Given the linearity of the wealth dynamics and the homogeneity of the utility function, the investor's problem can be written

$$F(W_t) = \max_{(c,z)} E_t \left[\int_t^\infty \frac{c_s^{1-b}}{1-b} e^{-\theta(s-t)} ds \right]$$

$$\text{s.t. } dW_s = (rW_s - c_s + z_s(\mu - r))\, ds + \sigma z_s dw_s$$
$$W_s \geq \alpha\underline{M},\ W_t > 0 \text{ given,}$$

where the parameters adjusted for growth are $r = r' - \lambda$, $\mu = \mu' - \lambda$, and the adjusted time discount rate $\theta = \theta' + (b-1)\lambda$ is assumed to be strictly positive. We still require $A > 0$ and in addition we make the following assumptions:

A1. The interest rate adjusted for inflation is positive: $r > 0$.

A2. The excess mean return of the risky asset is positive: $\mu - r > 0$.

Assumption A1. is required for feasibility. Assumption A2. is made for convenience and without loss of generality.

First of all, note that the value function F is increasing and concave[8] in W. Then, for $W \geq \alpha \underline{M}$, the Hamilton Jacobi Bellman (HJB) equation of this problem is

$$\theta F = \max_{(c,z)} \frac{c^{1-b}}{1-b} + (rW - c + z(\mu - r))\, F' + \frac{\sigma^2}{2} z^2 F''.$$

The optimality conditions are

$$\begin{aligned} c^* &= (F')^{-\frac{1}{b}} \\ z^* &= -\frac{(\mu - r)F'}{\sigma^2 F''}, \end{aligned}$$

and F satisfies the following non-linear ODE

$$\theta' F = \frac{b(F')^{\frac{b-1}{b}}}{1-b} + rWF' - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 \frac{(F')^2}{F''}. \tag{4}$$

Lemma 1 *The general solution of ODE (4) is such that*

$$W = A(F'(W))^{-\frac{1}{b}} + L_1(F'(W))^{\frac{\beta_1 - 1}{b}} + L_2(F'(W))^{\frac{\beta_2 - 1}{b}}, \tag{5}$$

where β_1 and β_2 are respectively the positive and negative roots of the quadratic

$$\frac{1}{2}\left(\frac{\mu - r}{b\sigma}\right)^2 x^2 + \left(\frac{1}{A} - r - \frac{1}{2}\left(\frac{\mu - r}{b\sigma}\right)^2\right) x = \frac{1}{A}, \tag{6}$$

and L_1 and L_2 are two constants to be determined.

Proof. See the Appendix. ■

Useful results $\beta_1 > 1$ and $1 - b - \beta_2 > 0$ are proved in the Appendix.

Boundary Condition at the Minimum Floor. At $W = \alpha \underline{M}$, we have

$$\alpha \underline{M} = A(F'(\alpha \underline{M}))^{-\frac{1}{b}} + L_1(F'(\alpha \underline{M}))^{\frac{\beta_1 - 1}{b}} + L_2(F'(\alpha \underline{M}))^{\frac{\beta_2 - 1}{b}}, \tag{7}$$

and in order not violate the constraint with some positive probability in a near future, stock holdings must be zero, which implies

$$A = (\beta_1 - 1)L_1(F'(\alpha \underline{M}))^{\frac{\beta_1}{b}} + (\beta_2 - 1)L_2(F'(\alpha \underline{M}))^{\frac{\beta_2}{b}}. \tag{8}$$

[8]The strict concavity of F comes from the fact that the utility function is strictly concave and the constraint is linear so that if W and $W'$are admissible wealth processes, then for all λ in $[0, 1]$, $\lambda W + (1 - \lambda)W'$ is also admissible.

When W is large, the constraint is equivalent to $W \geq 0$, so the solution is equivalent to the one for the unconstrained case, i.e. $F'(W) \underset{\infty}{\sim} A^b(W)^{-b}$. Since $\frac{\beta_2-1}{b} < -\frac{1}{b}$, we must have $L_2 = 0$. Combining relationships (7) and (8) yields

$$L_1 = \left(\frac{\alpha \underline{M}}{\beta_1}\right)^{\beta_1} \left(\frac{A}{\beta_1 - 1}\right)^{1-\beta_1} > 0.$$

Note that at $W = \alpha\underline{M}$, the wealth dynamics are deterministic as $dW_t = \left(r - \frac{\beta_1-1}{\beta_1 A}\right) W_t dt$. Since $r - \frac{\beta_1-1}{\beta_1 A} = \frac{1}{2}\left(\frac{\mu-r}{b\sigma}\right)^2 (\beta_1 - 1) > 0$, the wealth process bounces back upward after hitting the minimum floor.

2.2.1 Properties of the Optimal Allocations

The consumption-wealth ratio $\frac{c^*}{W}$ is given by

$$\frac{c^*}{W} = \frac{1}{A + L_1(F'(W))^{\frac{\beta_1}{b}}}.$$

It is increasing in wealth and always smaller than in the unconstrained case. The fraction of wealth invested in the stock is given by

$$\frac{z^*}{W} = \frac{\mu - r}{b\sigma^2}\left(1 - \beta_1 + \frac{\beta_1 A}{A + L_1(F'(W))^{\frac{\beta_1}{b}}}\right).$$

This ratio is monotonic (increasing) in wealth and always smaller than in the unconstrained case. The reason is the rise of the relative risk aversion of the lifetime utility in wealth since

$$-\frac{WF''}{F'} = b\left(1 + \frac{\beta_1 L_1(F'(W))^{\frac{\beta_1}{b}}}{A + (1-\beta_1)L_1(F'(W))^{\frac{\beta_1}{b}}}\right) > b.$$

At the floor $W = \alpha\underline{M}$, the relative risk aversion is infinite and consequently holdings in stock are zero. Note that the risky investment strategy is not of CPPI (that is, constant proportion portfolio insurance) type as proposed by Black and Perold [3] and optimally derived by Grossman and Zhou [21] for a stochastic floor. As wealth increases, lifetime utility relative risk aversion decreases and as wealth becomes very large, the effects of the constraint vanish: optimal allocations converge to the optimal unconstrained ones.

Our analysis has shown that in presence of a *fixed* minimum floor, hedging motives induce a reduction in consumption and risky investment and enhance risk aversion. In the next section, the investor's ability to control the minimum floor combined with a ratchet effect lead to quite different properties of the optimal strategy as stock holdings and consumption plans serve as wealth growth regulators.

2.3 Consumption-Portfolio Choice Problem under a Drawdown Constraint

As before define $c_t \equiv \widehat{c}_t e^{-\lambda t}$, $z_t \equiv \widehat{z}_t e^{-\lambda t}, W_t \equiv \widehat{W}_t e^{-\lambda t}$ and $M_t \equiv \widehat{M}_t e^{-\lambda t}$ so that the drawdown constraint (3) is equivalent to

$$M_t = \sup_{0 \leq s \leq t} \{M_0, W_s; 0 \leq s \leq t\}.$$

Choosing adapted consumption plan c and adapted risky investment strategy z, the agent aims at maximizing her lifetime utility

$$F(W_t, M_t) = \max_{(c,z)} E_t \left[\int_t^{\infty} \frac{c_s^{1-b}}{1-b} e^{-\theta(s-t)} ds \right]$$

$$\begin{array}{l} \text{s.t. } dW_s = (rW_s - c_s + z_s(\mu - r))\, ds + \sigma z_s dw_s \\ \quad W_s \geq \alpha M_s,\ W_t > 0, M_t > 0 \text{ given.} \end{array} \tag{9}$$

It is assumed that $\alpha \in (0,1]$ and θ, r, $\mu - r > 0$, and $A > 0$ are all positive with $A^{-1} = \frac{\theta}{b} + \frac{b-1}{b}\left(r + \frac{(\mu-r)^2}{2b\sigma^2}\right)$.

This preference specification intends to capture two aspects that we think are important for understanding investor behavior. The first aspect is loss aversion over wealth driven by ego considerations. The second feature we wish to take into account is the fact that as an investor becomes richer, she may psychologically commit part of the profits made to finance future expenditures such as a new house or early retirement and savor the thought of extra future consumption. A sudden significant loss in wealth may really hurt. We start the analysis by reviewing some useful properties of the maximum process M and the value function F.

2.3.1 Properties of the Maximum Process

PM1. As mentioned in Grossman and Zhou [21], M is a continuous increasing process and thus a finite variation process.

PM2. Denoting by $[X, Y]$ the quadratic covariation between processes X and Y, we have $d\,[M, W]_t = 0$ and $d\,[M, M]_t = 0$.

2.3.2 Properties of the Value Function

P1. F is strictly increasing in W, decreasing in M and strictly concave in (W, M).

P2. F is homogeneous of degree $1-b$ in (W, M).

Proof. See the Appendix. ■

Property **P2** implies that

$$F(W, M) = M^{1-b} f(u),$$

with $u = \frac{W}{M}$ and some smooth, concave and strictly increasing function f.

2.3.3 Derivation of the Value Function

Given the properties of the maximum process M, for $W \in (\alpha M, M)$, the HJB associated to the investor's program is

$$\theta F = \max_{(c,z)} \frac{c^{1-b}}{1-b} + (rW - c + z(\mu - r))\, F_1 + \frac{\sigma^2}{2} z^2 F_{11}. \tag{10}$$

The optimality conditions can be written

$$\begin{aligned} c^* &= M(f'(u))^{-\frac{1}{b}} \\ \frac{z^*}{W} &= -\frac{(\mu - r) f'(u)}{\sigma^2 u f''(u)}, \end{aligned}$$

and for $u \in (\alpha, 1)$ the (reduced) value function f satisfies the non-linear ODE

$$\theta f(u) = \frac{b(f'(u))^{\frac{b-1}{b}}}{1-b} + ruf'(u) - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 \frac{(f'(u))^2}{f''(u)}. \tag{11}$$

As shown in lemma 1, the general solution f of the ODE (11) is such that

$$u = A(f'(u))^{-\frac{1}{b}} + K_1(f'(u))^{\frac{\beta_1 - 1}{b}} + K_2(f'(u))^{\frac{\beta_2 - 1}{b}}, \tag{12}$$

where K_1 and K_2 are two constants to be determined. In the sequel, we find that $K_1 > 0$ and $K_2 < 0$.

Interpretation of the Solution The optimal wealth process is the sum of three terms:

$$W = AM(f'(u))^{-\frac{1}{b}} + K_1 M(f'(u))^{\frac{\beta_1 - 1}{b}} + K_2 M(f'(u))^{\frac{\beta_2 - 1}{b}}.$$

The first term is the one found for the Merton problem and indicates that wealth is used to finance consumption plans. The second term is positive and incorporates hedging motives as in the fixed minimum floor problem. This term can be related to portfolio insurance strategies involving simple options such as in Black and Perold [3] and Tepla [33]. Finally, the third term is also a hedging component: it is negative and regulates the growth rate of the wealth to mitigate the ratchet effect of the stochastic floor.

The next step is to derive the boundary conditions at $u = \alpha$ and $u = 1$.

2.3.4 Boundary Conditions

The boundary conditions are derived in the Appendix. In summary, at $u = \alpha$, holdings in the risky asset must be zero. At $u = 1$, the condition must ensure that the Hamilton Jacobi Bellman equation still holds. There are two possibilities depending on the parameters: either $F_2(M, M) = 0$ or holdings in the risky asset is set to zero, $z^*(1) = 0$. Whenever condition $F_2(M, M) = 0$ leads to a globally concave[9] value function F, this is the optimality condition. Otherwise, $z^*(1) = 0$ must be imposed. Denoting $Y = (f'(1))^{\frac{1}{b}}$ and $X = (f'(\alpha))^{\frac{1}{b}}$, the boundary conditions are

$$\begin{aligned}
\alpha X &= A + K_1 X^{\beta_1} + K_2 X^{\beta_2} \\
A &= (\beta_1 - 1)K_1 X^{\beta_1} + (\beta_2 - 1)K_2 X^{\beta_2} \\
Y &= A + K_1 Y^{\beta_1} + K_2 Y^{\beta_2} \\
A - (\beta_1 - 1)K_1 Y^{\beta_1} - (\beta_2 - 1)K_2 Y^{\beta_2} &= \max\left\{0, \frac{1 - b - \beta_1\beta_2}{b-1}(A_0 - Y)\right\} \quad b \neq 1,
\end{aligned}$$

where $A_0^{-1} = \frac{\theta + (b-1)r}{b}$ and when $b = 1$, $Y = A = A_0 = \frac{1}{\theta}$. The existence and uniqueness of the quadruple (K_1, K_2, X, Y) are shown in the Appendix and we find that $K_1 > 0$ and $K_2 < 0$.

The first three boundary conditions are the same whether $F_2(M, M) = 0$ or $z^*(1) = 0$ is imposed. For $b \neq 1$, when $F_2(M, M) = 0$ is imposed, the fourth condition is

$$A - (\beta_1 - 1)K_1 Y^{\beta_1} - (\beta_2 - 1)K_2 Y^{\beta_2} = \frac{1 - b - \beta_1\beta_2}{b-1}(A_0 - Y), \tag{13}$$

and for $b = 1$, $Y = A = A_0$, whereas when $z^*(1) = 0$ is imposed, the fourth condition is

$$A = (\beta_1 - 1)K_1 Y^{\beta_1} + (\beta_2 - 1)K_2 Y^{\beta_2}.$$

Let S denote the 4 by 4 non-linear system defined by the first three boundary conditions and equation (13).

[9] In the Appendix, we show that the value function F is globally concave iff $(\beta_1 - 1)K_1 Y^{\beta_1} + (\beta_2 - 1)K_2 Y^{\beta_2} \leq A$.

Proposition 1 *If the solution of system S is such that, whenever $b \geq 1$ ($b \leq 1$), $Y \leq A_0$ ($Y \geq A_0$), then $F_2(M,M) = 0$ is the optimality condition. If $\theta \leq r$, in fact $F_2(M,M) = 0$ is always optimal. Conversely, if $\theta > r$, then for $b \neq 1$, there exists a critical value $\alpha^* \in (0,1)$ such that for all $\alpha \in (0,\alpha^*)$, $F_2(M,M) = 0$ is optimal. At $\alpha = \alpha^*$, we have $F_2(M,M) = 0$ and $z^*(1) = 0$. Then, for all $\alpha \in (\alpha^*, 1]$, $z^*(1) = 0$ is optimal and wealth can never reach a new height.*

Proof. See the Appendix. ∎

As developed in more details in the sequel, the intuition behind the results of proposition 1. is the agent's willingness of mitigating the ratchet impact (and the irreversible associated cost) of the drawdown constraint. An individual who highly discounts future (θ large) would like to consume most of her current wealth. Increasing the minimum floor αM implies putting a cap on the amount of current wealth that can be used to finance current consumption, which may turns out be too costly for such an agent, especially when the drawdown coefficient α is large. In the sequel, we show that maintaining wealth within a fixed band $[\alpha M_0, M_0]$ has interesting implications for the optimal consumption process.

A more precise characterization of the cut off value α^* is available. At $\alpha = \alpha^*$, we have $Y = A_0$. Let X be the root of the equation

$$A = (\beta_1 - 1)K_1 X^{\beta_1} + (\beta_2 - 1)K_2 X^{\beta_2},$$

with

$$K_1 = \frac{\beta_2 A - (\beta_2 - 1)A_0}{\beta_1 - \beta_2} A_0^{-\beta_1} \text{ and } K_2 = \frac{-\beta_1 A + (\beta_1 - 1)A_0}{\beta_1 - \beta_2} A_0^{-\beta_2}.$$

Then the cut off value α^* is given by

$$\alpha^* = AX^{-1} + K_1 X^{\beta_1 - 1} + K_2 X^{\beta_2 - 1}.$$

For the logarithmic preferences case, $b = 1$, Proposition 1 implies that $F_2(M,M) = 0$ is always optimal since in this case solution Y of system S is equal to A_0. When $z^*(1) = 0$ is optimal, we show in the Appendix that we always have $Y \geq A_0$.

Finally, note that when the drawdown constraint binds $W = \alpha M$, the wealth dynamics are deterministic

$$dW_t = (r - \frac{1}{\alpha X})W_t dt.$$

In the Appendix, we establish that $r > \frac{1}{\alpha X}$ is always satisfied, so that after hitting the minimum floor the wealth process W remains above the minimum floor αM in the next instant. Having solved the HJB equation and determined the boundary conditions at $u = \alpha$ and $u = 1$, we now analyze the properties of the optimal allocations.

2.4 Properties of the Optimal Allocations

2.4.1 Consumption

Optimal consumption c^* is implicitly defined by the relationship

$$\frac{W}{M} = G\left(\frac{c^*}{M}\right), \tag{14}$$

where $G(x) = Ax + K_1 x^{1-\beta_1} + K_2 x^{1-\beta_2}$. Since $G' > 0$, the optimal consumption c^* is increasing in current wealth W. Then, recall that the consumption wealth ratio is given by $\frac{c^*}{W} = \frac{1}{u}\left(f'(u)\right)^{-\frac{1}{b}}$, so

$$\frac{\partial}{\partial u}\left(\frac{c^*}{W}\right) = \frac{1}{bu^2}\left(f'(u)\right)^{-\frac{1}{b}}\left(-\frac{uf''(u)}{f'(u)} - b\right) > 0,$$

since due to hedging motives, we establish in the sequel that $\frac{z^*}{W} < \frac{\mu - r}{b\sigma^2}$, which implies that the lifetime utility relative risk aversion $-\frac{uf''(u)}{f'(u)}$ is above its unconstrained level b.

The consumption-wealth ratio $\frac{c^*}{W}$ is increasing in the ratio current wealth over its peak $\frac{W}{M}$, so in particular increasing in current wealth W and decreasing in the historical maximum level of wealth M. At the ceiling $W = M$, we have $\frac{c^*}{M} = \frac{1}{Y}$. If $b > 1$, we have $Y \geq A_0 \geq A$ so we can conclude that for all u in $[\alpha, 1]$, $\frac{c^*}{W} < \frac{1}{A}$. Recall that the intertemporal elasticity of substitution (IES) s is equal to $\frac{1}{b}$. An investor who is reluctant ($s < 1$) to alter her consumption plans overtime chooses to consume a lower fraction of her wealth than she does in the unconstrained case. If $b < 1$ and $F_2(M, M) = 0$ is optimal, we have $Y < A$. An investor willing to alter her consumption plans ($s > 1$) has a consumption-wealth ratio that is larger than in the unconstrained case for large values of the ratio $\frac{W}{M}$. For α close to 1, this property is global[10] i.e. for all u in $[\alpha, 1]$, $\frac{c^*}{W} > \frac{1}{A}$. If $b < 1$ and $z^*(1) = 0$, both cases $\frac{1}{Y} > \frac{1}{A}$ and $\frac{1}{Y} < \frac{1}{A}$ are possible.

Next, we show that optimal consumption inherits a ratcheting behavior from wealth and habit formation arises endogenously.

All-Time High Consumption and Habit Formation. Denoting $c^*_{M_t} = \sup_{0 \leq s \leq t} \{c^*_s\}$ the maximum to date level of consumption, for $0 \leq s \leq t$, we have $\frac{1}{X} \leq \frac{c^*_s}{M_s} \leq \frac{1}{Y}$. Since $M_s \leq M_t$, it follows that for all date t,

$$\frac{Y}{X} \leq \frac{c^*_t}{c^*_{M_t}}.$$

The current consumption level c^*_t over its peak $c^*_{M_t}$ remains within the fixed band $[\alpha_c, 1]$, with $\alpha_c = \frac{Y}{X} < 1$. The maximum drawdown in consumption from its previous all-time high is $1 - \alpha_c$ and it decreases as α goes up (see the Appendix). Imposing ratcheting on the wealth process induces a ratcheting behavior of the optimal consumption as posited by Duesenberry [15] and analytically derived by Dybvig [16]. When the investor does not tolerate any decline in consumption, Dybvig establishes that for all times t, the wealth process W_t must optimally be maintained within the band $\left[\frac{c^*_{M_t}}{r}, \frac{(1-\beta_2)c^*_{M_t}}{-\beta_2 r}\right]$. This implies that current wealth W_t must be kept above the proportion $\frac{-\beta_2}{1-\beta_2}$ of its peak M_t. Grossman and Zhou [21] claim that the reason for such a restriction on the manager's investment policy is that the owner of the fund psychologically (and often physically) commits to use part of the profit when reaching the peak. Dybvig argues that imposing a drawdown constraint on wealth seems ad hoc from an economic point of view, and his motivation was to offer an alternative to the work by Grossman and Zhou [21]. Although the problem studied here and Dybvig's model are not equivalent, our analysis provides a bridge between the two approaches as well as an economic justification in terms of preferences (habit formation) over consumption for downside protection on wealth. The drawdown constraint (3) is a practical and effective way to ensure that standard of living will not have to be lowered by too much in the case of an adverse shock.

We now investigate the impact of the magnitude of the drawdown proportion α on the consumption-wealth ratio.

[10] In the limit case $\alpha = 1$, when $F_2(M, M) = 0$ is optimal, we have $X = Y = A_0$. Since for $b < 1$, $\frac{1}{A_0} > \frac{1}{A}$, by continuity in α, for α large enough, it follows that $\alpha X < A$, which implies that $\frac{c^*}{W} > \frac{1}{A}$, for all u in $[\alpha, 1]$.

Proposition 2 *If $z^*(1) = 0$ is optimal, the more stringent the drawdown constraint (higher α), the smaller the consumption-wealth ratio for all u in $[\alpha, 1]$. When $F_2(M, M) = 0$ is optimal, if $b \geq 1$, the previous result remains valid. However, if $b < 1$, there is a critical value u^*_α in $(\alpha, 1)$, such that the consumption-wealth ratio decreases in α on $[\alpha, u^*_\alpha]$ and increases on $[u^*_\alpha, 1]$.*

Proof. See the Appendix. ■

Proposition 2 suggests that for an investor with a high IES ($s > 1$), when wealth is about to reach its peak, for large values of α, the investor relies on the consumption margin to regulate the growth of her wealth and dampen the ratchet effect.

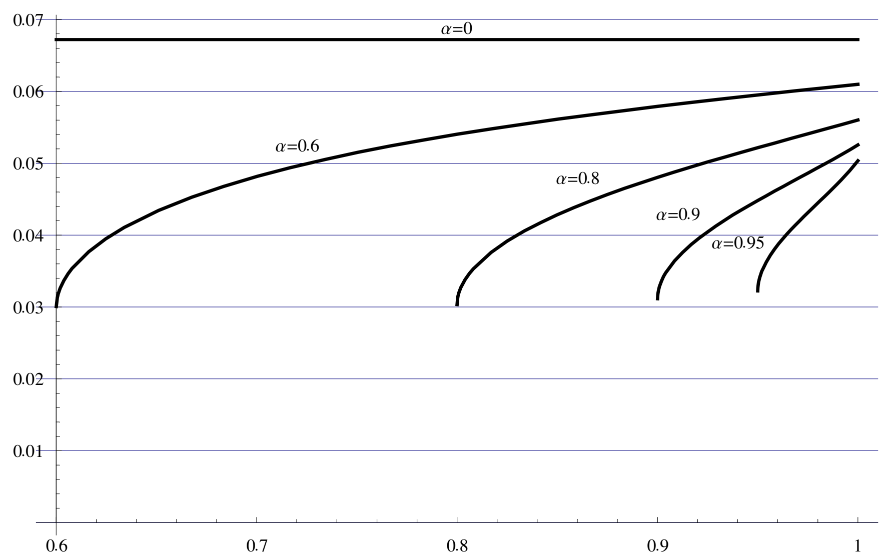

Figure 1 : Consumption-wealth ratio $\frac{c^*}{W}$ as a function of u

$\mu = 0.12,\ r = 0.04,\ \sigma = 0.2,\ \theta = 0.06,\ b = 2.5$

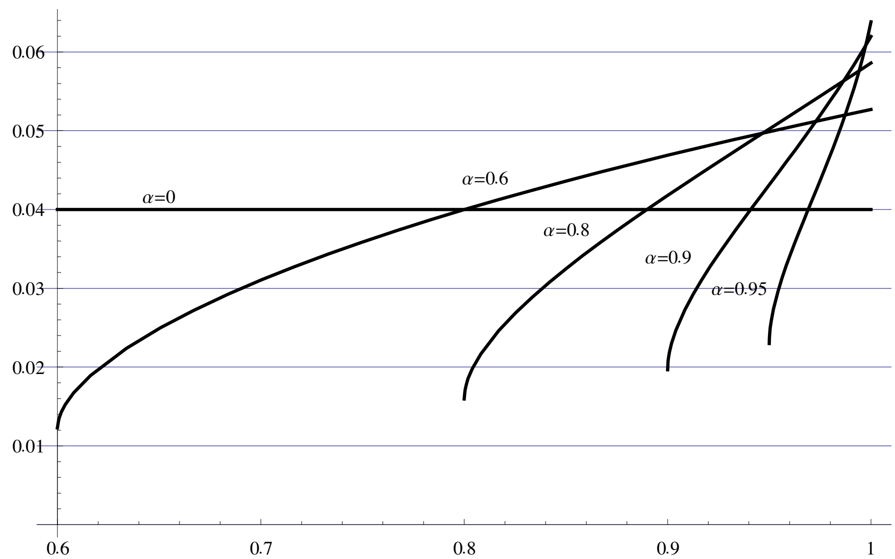

Figure 2 : Consumption-wealth ratio $\frac{c^*}{W}$ as a function of u

$\mu = 0.12,\ r = 0.04,\ \sigma = 0.2,\ \theta = 0.06,\ b = 0.8$

The consumption-wealth ratio $\frac{c^*}{W}$ is displayed in Figures 1 and 2 for several values of the drawdown constraint parameter α. In Figure 1, for $b > 1$, as α goes up, $\frac{c^*}{W}$ uniformly shrinks and remains below the unconstrained ratio $\frac{1}{A} = 0.0672$. The reduction in consumption is large when wealth is close the minimum floor. For $\alpha = 0.6$, 0.8, 0.9 and 0.95, the endogenous ratchet coefficient for consumption α_c is 0.29, 0.43, 0.53 and 0.61 respectively. In figure 2, for $b < 1$, curves cross with one another and as asserted in proposition 2 when u is high enough, an increase in α leads to a higher consumption-wealth ratio that significantly exceeds the unconstrained ratio $\frac{1}{A} = 0.04$. For $\alpha = 0.6$, 0.8, 0.9 and 0.95, the values obtained for α_c are 0.14, 0.22, 0.28 and 0.34 respectively. Observe that when $b < 1$, larger drawdowns $1 - \alpha_c$ from all-time high consumption level are allowed than in the case $b > 1$, reflecting the fact that individuals with a higher IES tolerate larger changes in their consumption plans across time.

2.4.2 Asset Allocations

The fraction of wealth invested in the risky asset is given by

$$\frac{z^*}{W} = \frac{\mu - r}{b\sigma^2}\left(1 - \frac{\beta_1 K_1 (f'(u))^{\frac{\beta_1}{b}} + \beta_2 K_2 (f'(u))^{\frac{\beta_2}{b}}}{A + K_1 (f'(u))^{\frac{\beta_1}{b}} + K_2 (f'(u))^{\frac{\beta_2}{b}}}\right). \tag{15}$$

The fraction of wealth invested in the risky asset is lower than in the unconstrained case, i.e. $\frac{\mu - r}{b\sigma^2}$. This is due in part to the hedging motives as described in the section 2.2. However, numerical simulations (displayed in the sequel) indicate that the investor's desire to dampen the ratchet effect plays a significant role in explaining the reduction in risky investment. Finally, note that the optimal risky investment strategy is not of the CPPI type as in Dybvig [16] where the demand for stocks is proportional to the excess of wealth over the perpetuity value of current consumption or as in

Grossman and Zhou [21] where risky investment is linear in the excess of wealth over the minimum stochastic floor (when $\lambda = 0$).

Proposition 3 *When $F_2(M,M) = 0$ is optimal, if $b > 1 (b < 1)$ and $\theta < \frac{1}{Y} (\theta > \frac{1}{Y})$, the fraction of wealth invested in the risky asset is non-decreasing in the ratio $\frac{W}{M}$; otherwise it is hump-shaped. When $z^*(1) = 0$ is optimal, the fraction of wealth invested in the stock and the ratio $\frac{W}{M}$ are linked by an inverted U-relationship.*

Proof. See the Appendix. ■

Conditions for the logarithmic investor are more cumbersome and are presented in the Appendix.

Proposition 3 deserves several observations. First, raising the fraction of wealth invested in stocks when wealth goes up is optimal only if the cost associated with the ratchet effect is not too large. Observe that θ is the consumption-wealth ratio at $u = 1$ for the myopic investor ($b = 1$). When $b > 1$, the investor's IES is low ($s < 1$) and she is mainly concerned with the current consumption-wealth ratio and is reluctant to defer consumption. Proposition 3 suggests that the agent optimally chooses an increasing risky investment policy provided that at $u = 1$, $\frac{c^*}{M} = \frac{1}{Y}$ is above the corresponding value for the myopic investor. Conversely, if $b < 1$, the individual is eager to defer consumption and accepts a low level of her current consumption-wealth ratio (below that of the myopic investor) at $u = 1$; consequently the fraction of wealth invested in stocks is increasing.

Second, decreasing stock holdings as a percentage of wealth when W is close to M depart from the results obtained in Grossman and Zhou [21] where the fraction of wealth invested in stock always increases in the ratio $\frac{W}{M}$. Recall that in Grossman and Zhou [21] there is no intermediate consumption so intertemporal consumption substitution plays no role. Nevertheless, the hump-shaped relationship corroborates the intuition pointed out by these authors, i.e. αM is expected to grow at a faster rate than W and therefore investment in the risky asset is expected to fall. The lifetime utility relative risk aversion $-\frac{uf''(u)}{f'(u)}$ is no longer decreasing as (current) wealth rises but instead is U-shaped.

The condition for the ratio $\frac{z}{W}$ to be non-decreasing in $\frac{W}{M}$ depends on all the parameters of the model. When $F_2(M,M) = 0$ is optimal, we establish in the Appendix that a raise in the drawdown coefficient α increases (decreases) Y whenever $b > 1 (b < 1)$ and $\lim_{\alpha \to 1} Y = A_0$. From Proposition 3, we deduce that a sufficient condition for $\frac{z}{W}$ to be non-decreasing in $\frac{W}{M}$ is simply $\theta < r$, i.e. the investor must be patient enough.

Proposition 4 *The more stringent the drawdown constraint (higher α), the smaller the fraction of wealth invested in the risky asset.*

Proof. See the Appendix. ■

Proposition 4 formally states that when the drawdown constraint becomes more stringent the fraction of wealth invested in the stocks $\frac{z^*}{W}$ is *uniformly* reduced, suggesting that indeed risky investment is the favored channel to achieve wealth growth regulation.

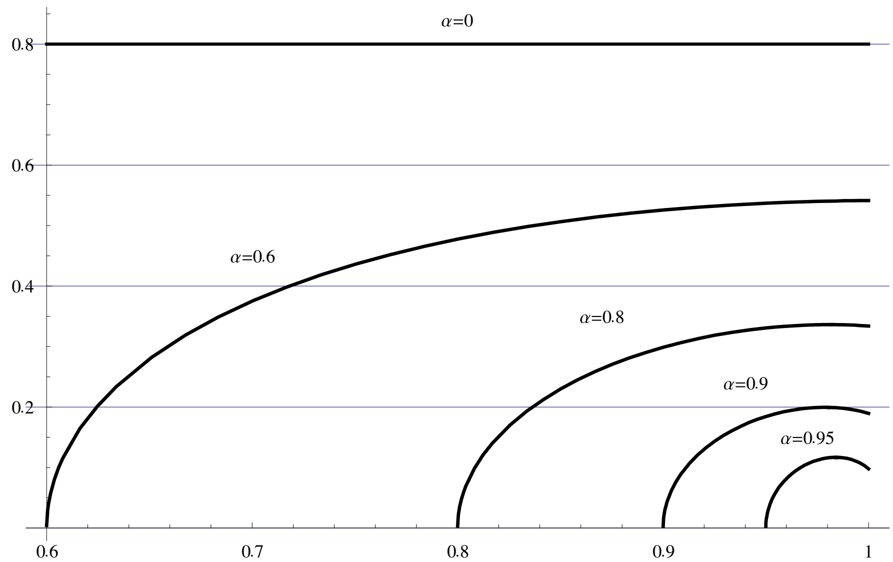

Figure 3 : Fraction of wealth invested in stocks $\frac{z^*}{W}$ as a function of u

$\mu = 0.12,\ r = 0.04,\ \sigma = 0.2,\ \theta = 0.06,\ b = 2.5$

Figure 3 depicts the fraction of wealth invested in the risky asset $\frac{z^*}{W}$ for several values of the drawdown constraint parameter α. As α goes up, risky investment is reduced and when α is large enough, the curve $\frac{z^*}{W}$ is hump shaped. As a benchmark, when $\alpha = 0$, the unconstrained allocation the fraction $\frac{\mu - r}{b\sigma^2} = 0.8$. Indeed, observe that even when the current wealth W_t is far from the minimum floor αM_t, the reduction in stock holdings can be substantial.

Obviously, the analysis performed combined both hedging and ratchet effects. In order to disentangle the two effects, consider a fixed minimum floor equal to αM and compute the fraction of wealth invested in the stock $\frac{\bar{z}^*}{W}$ when wealth W varies from αM up to M. Note that the ratio $\frac{\bar{z}^*}{W}$ is independent of the choice of M.

Table I: Disentangling hedging and ratchet effects

α	u	0.6	0.65	0.7	0.75	0.8	0.85	0.9	0.925	0.95	0.975	1
0.6	$\frac{\bar{z}^*}{W}$	0	0.283	0.383	0.450	0.499	0.537	0.568	0.581	0.592	0.603	0.613
	$\frac{z^*}{W}$	0	0.279	0.375	0.436	0.478	0.506	0.526	0.532	0.537	0.540	0.541
0.8	$\frac{\bar{z}^*}{W}$	-	-	-	-	0	0.248	0.339	0.372	0.402	0.428	0.450
	$\frac{z^*}{W}$	-	-	-	-	0	0.230	0.299	0.318	0.331	0.336	0.334
0.9	$\frac{\bar{z}^*}{W}$	-	-	-	-	-	-	0	0.168	0.234	0.283	0.322
	$\frac{z^*}{W}$	-	-	-	-	-	-	0	0.143	0.184	0.199	0.190

Table I reports stock holdings $\frac{\bar{z}^*}{W}$ and those corresponding to the drawdown problem $\frac{z^*}{W}$ for several values of α. Recall that for the Merton Problem, this ratio is constant and equal to 0.8. Observe that hedging motives explain a significant share of the reduction in risky investment. Nevertheless, the ratchet effect becomes significant when the ratio $\frac{W}{M}$ approaches 1 and is enhanced as the drawdown constraint becomes more stringent (higher α). Taking the unconstrained portfolio allocation as a benchmark, at $u = 1$, the ratchet effect accounts for 9%, 14.5% and 16% for $\alpha = 0.6$, 0.8 and 0.9 respectively of the total reduction in stock holdings.

2.4.3 Representation of the Optimal Wealth Process

The optimal policy (c^*, z^*) has been expressed in terms of state variables (W, M) using dynamic programming. Alternatively, it is possible to provide a representation in terms of simple regulated stochastic processes and gain some insights about the dynamics across time. Details of the derivation are presented in the Appendix.

Recall that $\frac{W}{M} = G\left(\frac{c^*}{M}\right)$, so that $\frac{c^*}{M} = H\left(\frac{W}{M}\right)$, with $H = G^{-1}$. First, we establish that the process $\frac{c^*}{M}$ is a two sided regulated geometric Brownian motion[11] with lower barrier $\frac{1}{X}$ and upper barrier $\frac{1}{Y}$ and for u in $(\alpha, 1)$, the dynamics are given by

$$d\left(\frac{c_t^*}{M_t}\right) = \frac{c_t^*}{M_t}\left((r - \frac{1}{A} + \frac{(\mu - r)^2}{b\sigma^2})dt + \frac{\mu - r}{b\sigma}dw_t\right).$$

Observe that this law of motion is the same as the one that governs the optimal consumption process in the Merton problem (section 2.1). Second, the representation of current wealth W, all time maximum wealth M and consumption c^* as stochastic processes depends on the boundary condition at $u = 1$. When $F_2(M, M) = 0$ is optimal, we show that the process $\log H\left(\frac{W_t}{M_t}\right)$ is a one sided regulated arithmetic Brownian motion with lower barrier $-\log Y$, and for u in $(\alpha, 1)$,

$$d\log H\left(\frac{W_t}{M_t}\right) = (\frac{r - \theta}{b} + \frac{(\mu - r)^2}{2b\sigma^2})dt + \frac{\mu - r}{b\sigma}dw_t.$$

If $z^*(1) = 0$ is optimal, we establish in proposition 1 that wealth can never exceed its (initial) all time high level M_0. The ceiling $W = M$ is an upper reflecting barrier; the optimal consumption process c^* is a two sided regulated geometric Brownian motion. Next, we estimate the cost induced by the constraint.

2.5 Cost of the Drawdown Constraint

There are several ways of estimating the cost of the drawdown constraint. We can assess the loss in terms of forgone lifetime utility; alternatively, we can measure it in terms of the numéraire. We start with the first measure and to keep things simple, we derive the maximum cost when $\alpha = 1$.

2.5.1 Cost in Terms of Forgone Lifetime Utility

Wealth must always be maintained at its maximum, so in order not to violate the drawdown constraint, holdings in the stock must be zero $z^* \equiv 0$. It is easy to verify that if $r \geq \theta$, the optimal solution is $c^* = \frac{W}{A_0}$ and wealth dynamics are deterministic with $dW_t = \frac{r-\theta}{b}W_t dt \geq 0$. Conversely, if $r \leq \theta$, we have a corner solution with $c^* = rW$ and $dW_t = 0$: wealth remains at its initial level W_0. The corresponding value function is $F_0(W) = \frac{K^b W^{1-b}}{1-b}$, with

$$K = \begin{cases} A_0, \text{ if } r \geq \theta \\ r^{\frac{1-b}{b}}\theta^{-\frac{1}{b}}, \text{ if } r \leq \theta. \end{cases}$$

[11] For the definition of a regulated Brownian motion, see Harrison [23], p14.

The (maximum) cost of the drawdown constraint in terms of loss of the lifetime utility is the relative difference between the constrained and unconstrained value functions. When $b \neq 1$, it is simply given by

$$\left(\frac{K}{A}\right)^{\varepsilon b} - 1,$$

where $\varepsilon = 1(\varepsilon = -1)$ if $b > 1(b < 1)$. For $\mu = 0.12$, $\sigma = 0.2$, $\theta = 0.05$, $b = 2.5$, the loss is approximately 55.4% when $r = 0.06$. It decreases with the instantaneous variance σ^2 but increases with the mean return μ. When $r = 0.04$, the loss is approximately 151%, which illustrates that imposing a drawdown constraint severely penalizes impatient individuals.

2.5.2 Cost in Terms of the Numéraire

We calculate the percentage k increase in wealth necessary to bring the level of the lifetime utility to the level of those of an unconstrained investor, i.e. we want to determine k such that $F((1+k)W) = \frac{A^b}{1-b}W^{1-b}$. We obtain

$$k = \left(\frac{K}{A}\right)^{\frac{b}{1-b}} - 1,$$

and for the parameters chosen previously, we find that the percentage increase k is approximately 34.2% when $r = 0.06$. It also decreases with the instantaneous variance σ^2 and increases with the mean return μ. When $r = 0.04$, the necessary percent increase k is 84.7%, again reflecting the high cost of the constraint for impatient individuals.

For both measures used, the cost induced by the constraint is economically significant.

3 EXTENSION OF THE BASIC MODEL

In this section, we consider the case of an agent who derives utility from current consumption and also from her status. There are two rival theories of social status: ascription versus achievement. Individual position can be ascribed by virtue of their age, sex, race, and family membership or connection. Alternatively, individuals can achieve their own position by their own performance and merits. Here, we interpret a society in which higher wealth confers a higher status. Status can confer power, privileges, access to political circles or social events, and at a more personal level, enhance self esteem. As argued in Cole, Mailath, and Postlewaite [7], social status can determine the degree of success one group member may have with non-market decisions such as finding a good mate for instance. Weber [37] refers to a status group as a collection of individuals who happen to have a common lifestyle and share the same economic interest. Maintaining one's membership of a status group is certainly desirable and ambition may dictate social climbing; however, individuals may be reluctant to lower their position in society. People who experience a downward social shift may experience depression or poor psychological well being.[12]

Following Bakshi and Chen [1] and Smith [31], to keep things simple, we retain current wealth level W as an index of status and status seeking is modeled as direct preference for financial wealth.

[12]The University of Newcastle upon Tyne study by Parker, Pearce and Tiffin [29] indicates that women are twice as likely to be downwardly mobile. The study involved men and women born in 1947 in Newcastle and followed them from childhood to age 50. Researchers noted the findings might be explained by the fact that men born during that era gained much of their self-esteem from their careers, whereas women found fulfillment from social pursuits outside of work, such as children and friendships.

More specifically, preferences[13] are given by

$$u(c, W, M) = \begin{cases} \frac{(c+aW)^{1-b}}{1-b}, & W \geq \alpha M \\ -\infty, & \text{otherwise,} \end{cases}$$

where parameter $a > 0$ governs how much the agent cares about her social status.

We first examine the optimal consumption-portfolio choices for an unconstrained investor.

3.1 Benchmark Case

In the absence of status downfall fear, for $W_t > 0$ given, the agent aims at maximizing her lifetime utility

$$F(W_t) = \max_{(c,z)} E_t \left[\int_t^\infty \frac{(c_s + aW_s)^{1-b}}{1-b} e^{-\theta(s-t)} ds \right]$$

$$\text{s.t. } dW_s = (rW_s - c_s + z_s(\mu - r))\, ds + \sigma z_s dw_s.$$

This optimization problem can be rewritten

$$F(W_t) = \max_{(\overline{c},z)} E_t \left[\int_t^\infty \frac{(\overline{c}_s)^{1-b}}{1-b} e^{-\theta(s-t)} ds \right]$$

$$\text{s.t. } dW_s = ((r+a)W_s - \overline{c}_s + z_s(\mu + a - (r+a)))ds + \sigma z_s dw_s,$$

with $\overline{c}_s = c_s + aW_s$. Clearly, the problem can be nested in a standard Merton problem as in section 2.1 with a riskfree rate $r + a$ and a mean return of the stock $\mu + a$. Optimal allocations are given by

$$\begin{aligned} \frac{c^*}{W} &= \frac{1}{\widetilde{A}} - a \\ \frac{z^*}{W} &= \frac{\mu - r}{b\sigma^2}, \end{aligned}$$

provided that $(\widetilde{A})^{-1} = \frac{\theta}{b} + \frac{b-1}{b}\left(r + a + \frac{(\mu-r)^2}{2b\sigma^2}\right) > 0$.

For our choice of the functional form of the utility function, status seeking only affects the optimal consumption plans. Recall that the agent derives utility through two channels: current consumption c and current wealth W. These two channels compete with each other: the higher the consumption, the lower wealth accumulation and therefore the lower the future status. An increase in status enjoyment (higher a) leads to a decrease in the consumption-wealth ratio: the agent chooses to foster wealth accumulation, which is reflected by a higher mean growth of the wealth process.

We now study the case when the agent is reluctant to accept large status downfalls.

3.2 Maintaining Social Status

It is easy to realize that the analysis performed in section 2.3 for the case $a = 0$ still applies if we substitute $\frac{1}{A}$ with $\frac{1}{\widetilde{A}}$ and replace (μ, r) with $(\mu + a, r + a)$ in the definition of roots β_1 and β_2.

We now investigate the quantitative impact of status on the optimal allocations.

[13]The choice of the functional form of the utility function is motivated by tractability reasons.

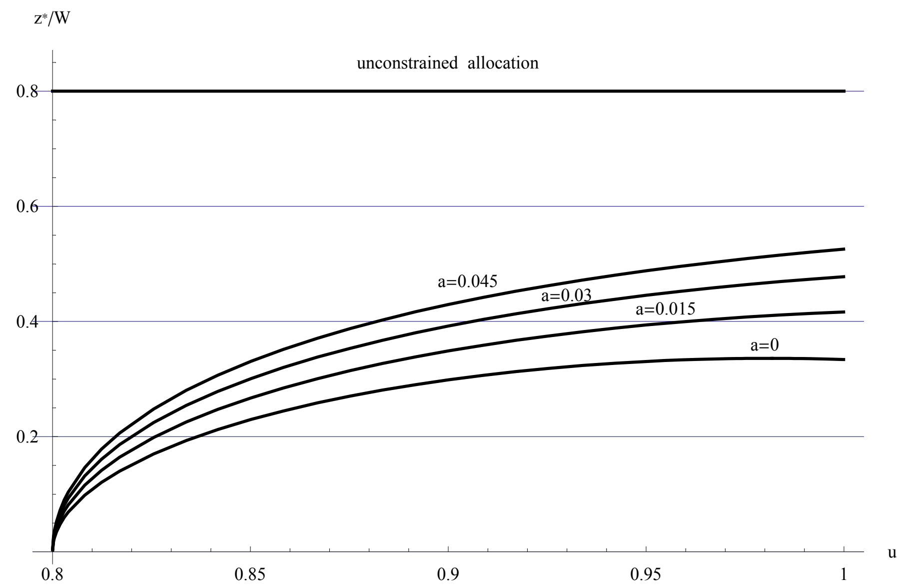

Figure 4 : Fraction of wealth invested in stocks $\frac{z^*}{W}$ as a function of u

$\mu = 0.12,\ r = 0.04,\ \sigma = 0.2,\ \theta = 0.06,\ b = 2.5,\ \alpha = 0.8$

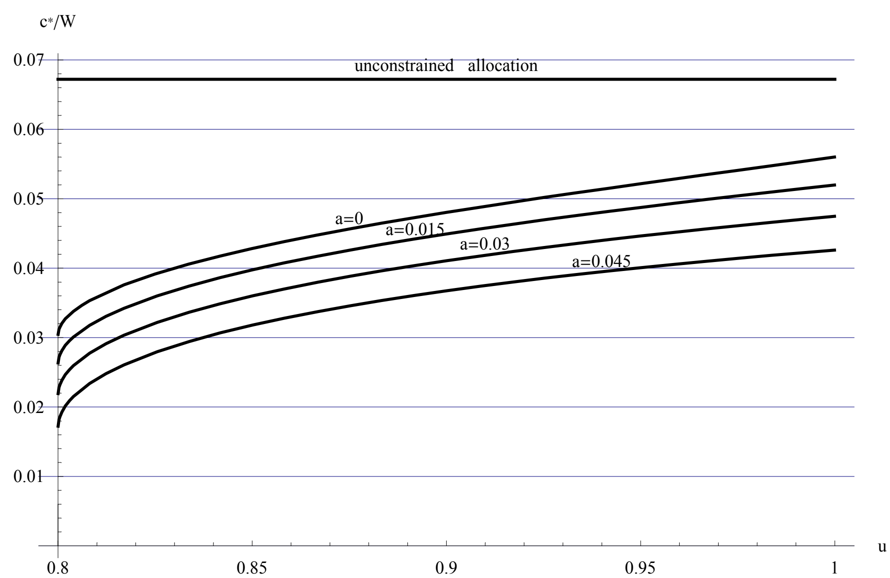

Figure 5 : Consumption-wealth ratio $\frac{c^*}{W}$ as a function of u

$\mu = 0.12,\ r = 0.04,\ \sigma = 0.2,\ \theta = 0.06,\ b = 2.5,\ \alpha = 0.8$

Figures 4 and 5 represent the fraction of wealth invested in the risky asset $\frac{z^*}{W}$ and the consumption-wealth ratio $\frac{c^*}{W}$ for several values of the parameter a. As a goes up, the investor increasingly values social status, which leads to heavier stock holdings in an attempt to achieve a higher growth rate of her wealth (Figure 4). In parallel, observe in Figure 5 that the consumption-wealth ratio uniformly shrinks. Priority shifts towards building up status that has a persistent impact whereas consumption is less appealing since its effect is only instantaneous. When a ratcheting behavior is imposed on wealth, introducing wealth as a proxy for social status in the utility function fosters risky investment in a substantial manner.

4 THE VERIFICATION THEOREM

The goal of this section is to formally establish that the heuristic proposed optimal strategy (c^*, z^*) is indeed valid. We closely follow Dybvig [16]. The proof consists of four steps.

For any feasible strategy (c, z), let define the process

$$M_t^b = \int_0^t u(c_s)e^{-\theta s}ds + e^{-\theta t}V(W_t, M_t),$$

where V denotes the value function that corresponds the feasible strategy (c, z).

Step 1: M^b is a local supermartingale for all admissible strategies (c, z) and a local martingale for the (proposed) optimal strategy (c^*, z^*).

There are three regions to examine. First, at $W_t = \alpha M_t$, any admissible strategy should be such $z_t = 0$ (otherwise the drawdown constraint would be violated with a positive probability in the next instant $t + h$, $h > 0$) and it follows that

$$\frac{dM_t^b}{e^{-\theta t}} = (u(c_t) - \theta V + \frac{1}{2}\sigma^2 z_t^2 V_{11})dt + V_1 dW_t + V_2 dM_t.$$

Since $z_t = 0$ and $dM_t = 0$ (as $W_t < M_t$), we have

$$\frac{E_t\left[dM_t^b\right]}{e^{-\theta t}} = (rW_tV_1 - \theta V)dt + (u(c_t) - c_tV_1)dt.$$

Bellman's principle states that

$$0 = \sup_{(c_t,z_t)} u(c_t) - \theta V + rW_tV_1 + (\mu - r)\, z_tV_1 + \frac{1}{2}\sigma^2 z_t^2 V_{11}.$$

Furthermore, since $V_1 > 0$ and u is strictly concave, for the proposed optimal strategy c^*, we have $u'(c_t^*) = V_1$, and it follows that $rWV_1 - \theta V + u(c_t^*) - c_t^*V_1 = 0$. Since it is a maximum, for all other admissible strategies c, we must have $rWV_1 - \theta V + u(c_t) - c_tV_1 \leq 0$, which implies that M^b is a local martingale for the (proposed) optimal strategy (c^*, z^*) and a local supermartingale for all feasible strategies (c, z). Second, for $\alpha M_t < W_t < M_t$, we have

$$\frac{E_t\left[dM_t^b\right]}{e^{-\theta t}} = (rW_tV_1 - \theta V + u(c_t) - c_tV_1 + (\mu - r)\, z_tV_1 + \frac{1}{2}\sigma^2 z_t^2 V_{11})dt.$$

Since for the (proposed) optimal strategy (c^*, z^*) $V_{11} < 0$, risky investment z_t^* is such that $z_t^* = -\frac{(\mu-r)V_1}{\sigma^2 V_{11}}$ and as before $u'(c_t^*) = V_1$. Since it is the maximum, all other possible strategies must have a non-positive drift, meaning that M^b is a local supermartingale. The drift is exactly zero for the optimal proposed strategy, so in this case M^b is a local martingale. Third, at $W_t = M_t$, using Ito lemma for semi-martingales, we have

$$\frac{E_t\left[dM_t^b\right]}{e^{-\theta t}} = (rW_tV_1 - \theta V + u(c_t) - c_tV_1 + (\mu - r)\, z_tV_1 + \frac{1}{2}\sigma^2 z_t^2 V_{11})dt + E_t\left[V_2 dM_t\right],$$

since M is an absolutely continuous process. For $h > 0$, over the interval of time $[t, t+h]$, the Bellman equation is

$$V(W_t, M_t) = \max_{(c_t,z_t)} E_t\left[u(c_t)h + e^{-\theta h}V(W_{t+h}, M_{t+h})\right],$$

again applying Ito lemma for semi-martingales, we have

$$\begin{aligned} 0 &= \max_{(c_t,z_t)} u(c_t)h + E_t\left[\int_t^{t+h}\left(-\theta V + (rW_s - c_s + z_s(\mu - r))\, V_1 + \frac{1}{2}\sigma^2 z_s^2 V_{11}\right) ds\right] \\ &\quad + E_t\left[\int_t^{t+h} V_2 dM_s\right]. \end{aligned}$$

Since $E_t\left[M_{t+h} - M_t \mid W_t = M_t\right] = \sqrt{\frac{2}{\pi}}\sigma\left|z_t\right|\sqrt{h} + O(h)$, when h is small, $\sqrt{h}$ dominates h so in order for the Bellman equation to hold at $W_t = M_t$, for the (proposed) optimal strategy (c^*, z^*) we must have $F_2(M_t, M_t) = 0$ or $z_t^* = 0$. In the latter case, we establish in the Appendix that wealth never achieves a new maximum so $dM_t = 0$. Thus, for strategy (c^*, z^*), at $W_t = M_t$, we always have $V_2 dM_t = 0$. Furthermore, by construction, as within the second region, in this case, the drift of dM_t^b is exactly zero, which shows that M^b is a local martingale for the (proposed) optimal strategy (c^*, z^*). Since it is a maximum, for all other feasible strategies the drift must be non-positive

$$rWV_1 - \theta V + u(c_t) - c_tV_1 + (\mu - r)\, z_tV_1 + \frac{1}{2}\sigma^2 z_t^2 V_{11} + E_t\left[V_2 dM_t\right] \leq 0.$$

We can conclude that M_t^b is a local supermartingale for all admissible strategies.

Step 2: M^b is supermartingale for all feasible strategies (c, z) and a martingale for the optimal (proposed) strategy (c^*, z^*).

Given some initial condition (W_0, M_0), for any admissible strategy, the corresponding value function cannot be smaller than the one obtained with the same admissible strategy but with initial condition $(\alpha M_0, M_0)$. At $W_t = \alpha M_t$, the wealth dynamics are deterministic

$$dW_t = (rW_t - c_t)dt.$$

Choosing $c_t \equiv \frac{W_t}{r}$ induces that $dW_t \equiv 0$ and for $t \geq 0$, we have $W_t = \alpha M_0$. Such a strategy is the worst possible and leads to the value function $V(\alpha M_0, M_0) = \frac{u(\frac{\alpha M_0}{r})}{\theta}$. It follows that

$$\begin{aligned} M_t^b &\geq \int_0^t u(\frac{\alpha M_0}{r})e^{-\theta s}ds + e^{-\theta t}\frac{u(\frac{\alpha M_0}{r})}{\theta} \\ &= \frac{u(\frac{\alpha M_0}{r})}{\theta}. \end{aligned}$$

This implies that M^b is a supermartingale because a local supermartingale bounded from below is a supermartingale. Then, to show that for the optimal proposed strategy, M^b is a martingale, we follow the proof provided in Dybvig [16]. Lemma 4. in Dybvig [16] p307 remains valid for our proposed optimal strategy since the wealth process W is **bounded away from** 0 as all $t \geq 0$, $W_t \geq \alpha M_0 > 0$ and for the proposed optimal strategy the fraction of wealth invested in the risky asset $\frac{z_s}{W_s}$ is in the compact set $\left[0, \frac{\mu - r}{b\sigma^2}\right]$. Furthermore, notice that Lemma 5 in Dybvig [16] p307 also remains valid. Combining both Lemmas, we can conclude that M^b is a martingale for the proposed optimal strategy.

Step 3: Asymptotic behavior of the residual term $e^{-\theta t} V(W_t, M_t)$.

For all feasible strategies $V(W_t, M_t) \geq \frac{u(\frac{\alpha M_0}{r})}{\theta}$, which implies that for all $b > 0$

$$\liminf_{t\to\infty} \ V(W_t, M_t) e^{-\theta t} \geq \lim_{t\to\infty} e^{-\theta t} \frac{u(\frac{\alpha M_0}{r})}{\theta} \geq 0.$$

For $b > 1$, for *all* feasible strategies, the value function V is always negative, so the previous inequality implies that

$$\lim_{t\to\infty} V(W_t, M_t) e^{-\theta t} = 0.$$

Then, for $b < 1$, differentiating relationship (12) with respect to recall u yields

$$\frac{f'(u)}{f''(u)} = -\frac{A}{b}(f'(u))^{-\frac{1}{b}} + \frac{\beta_1 - 1}{b} K_1 (f'(u))^{\frac{\beta_1 - 1}{b}} + \frac{\beta_2 - 1}{b} K_2 (f'(u))^{\frac{\beta_2 - 1}{b}}, \tag{16}$$

Since $f'' < 0$, combining relationships (12) and (16) leads to

$$\begin{aligned} \beta_1 A f'(u)^{-\frac{1}{b}} &\geq (\beta_2 - \beta_1) K_2 (f'(u))^{\frac{\beta_2 - 1}{b}} + (\beta_2 - 1) u \\ \beta_2 A f'(u)^{-\frac{1}{b}} &\geq (\beta_1 - \beta_2) K_1 (f'(u))^{\frac{\beta_1 - 1}{b}} + (\beta_1 - 1) u, \end{aligned}$$

for $u \in [\alpha, 1]$. This implies that

$$A^b \left(\frac{\beta_2}{\beta_2 - 1}\right)^b u^{-b} \leq f'(u) \leq A^b \left(\frac{\beta_1}{\beta_1 - 1}\right)^b u^{-b}. \tag{17}$$

From the (reduced) HJB equation (11), using relationship (16) we find that

$$\theta f(u) \leq \frac{b(f'(u))^{\frac{b-1}{b}}}{1 - b} + r u f'(u) + \frac{1}{2b}\left(\frac{\mu - r}{\sigma}\right)^2 A (f'(u))^{\frac{b-1}{b}}.$$

Using relationship (17), we obtain that for $u \in [\alpha, 1]$, $f(u) \leq K u^{1-b}$, where

$$K = \frac{b}{|1 - b|\,\theta}\left(1 + \frac{|1 - b|}{2}\left(\frac{\mu - r}{b\sigma}\right)^2 A\right) A^{b-1}\left(\frac{\beta_2}{\beta_2 - 1}\right)^{b-1} + \frac{r}{\theta} A^b \left(\frac{\beta_1}{\beta_1 - 1}\right)^b > 0$$

is a constant, and we have $F(W, M) \leq K W^{1-b}$. For the case $b = 1$, the HJB equation is

$$\theta f(u) = -\ln f'(u) - 1 + r u f'(u) - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 \frac{(f'(u))^2}{f''(u)},$$

and again using relationships (16) and (17), we obtain that for all $u \in [\alpha, 1]$, $f(u) \leq \frac{\ln u}{\theta} + L$, where

$$L = \frac{\ln\theta - 1 + \ln\frac{\beta_2 - 1}{\beta_2} + \frac{r}{\theta}\frac{\beta_1 - 1}{\beta_1} + \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 A}{\theta}$$

is a constant, and we have $F(W,M) \leq \frac{\ln W}{\theta} + L$. In addition, recall that for any feasible strategy $\frac{u(\frac{\alpha M_0}{r})}{\theta} \leq V(W,M)$. Following the same steps as Dybvig [16] in lemma 6 p308, it follows that, for $b \leq 1$

$$\lim_{t\to\infty} F(W_t, M_t)e^{-\theta t} = 0.$$

Step 4: Optimality of the proposed solution (c^*, z^*)**.**

Assumption (1) implies that a consumption plan c that satisfies $E_0\left[\int_0^\infty u(c_s)e^{-\theta s}ds\right] < \infty$ must be such that $\int_0^\infty u(c_s)^+ e^{-\theta s}ds < \infty$ and $\int_0^\infty u(c_s)^- e^{-\theta s}ds < \infty$ (almost surely). Then, for all $t \geq 0$, we have $\int_0^t u(c_s)^+ e^{-\theta s}ds < \int_0^\infty u(c_s)^+ e^{-\theta s}ds$, and by assumption $E_0\left[\int_0^\infty u(c_s)^+ e^{-\theta s}ds\right] < \infty$. Using Lebesgue Monotone Convergence Theorem, we obtain that

$$\lim_{t\to\infty} E_0\left[\int_0^t u(c_s)^+ e^{-\theta s}ds\right] = E_0\left[\int_0^\infty u(c_s)^+ e^{-\theta s}ds\right].$$

The same property holds for the negative part u^- of the utility function u. Finally since $u(c) = u(c)^+ - u(c)^-$, it follows that

$$\lim_{t\to\infty} E_0\left[\int_0^t u(c_s)e^{-\theta s}ds\right] = E_0\left[\int_0^\infty u(c_s)e^{-\theta s}ds\right].$$

For the proposed optimal strategy (c^*, z^*), M^b is a martingale and $\lim_{t\to\infty} E_0\left[e^{-\theta t}F(W_t, M_t)\right] = 0$. Therefore

$$\begin{aligned}
F(W_0, M_0) &= M_0^b \\
&= E_0\left[M_t^b\right] \text{ for all times } t \geq 0 \\
&= \lim_{t\to\infty} E_0\left[M_t^b\right] \\
&= \lim_{t\to\infty} E_0\left[\int_0^t u(c_s)e^{-\theta s}ds + e^{-\theta t}F(W_t, M_t)\right] \\
&= \lim_{t\to\infty} E_0\left[\int_0^t u(c_s)e^{-\theta s}ds\right] + \lim_{t\to\infty} E_0\left[e^{-\theta t}F(W_t, M_t)\right] \\
&= E_0\left[\int_0^\infty u(c_s)e^{-\theta s}ds\right],
\end{aligned}$$

where exchanging the expectation and the limit is justified by $u(c_s) \geq u(\frac{M_s}{X}) \geq u(\frac{M_0}{X})$. For any feasible strategy (c, z), M^b is a supermartingale and $\lim_{t\to\infty} \inf\ E_0\left[e^{-\theta t}V(W_t, M_t)\right] \geq 0$. Therefore

$$\begin{aligned}
V(W_0, M_0) &= M_0^b \\
&\geq E_0\left[M_t^b\right] \text{ for all times } t \geq 0 \\
&\geq \lim_{t\to\infty}\ E_0\left[M_t^b\right] \\
&= \lim_{t\to\infty} E_0\left[\int_0^t u(c_s)e^{-\theta s}ds + e^{-\theta t}V(W_t, M_t)\right] \\
&= \lim_{t\to\infty} E_0\left[\int_0^t u(c_s)e^{-\theta s}ds\right] + \lim_{t\to\infty} E_0\left[e^{-\theta t}V(W_t, M_t)\right] \\
&\geq E_0\left[\int_0^\infty u(c_s)e^{-\theta s}ds\right].
\end{aligned}$$

This completes the proof. ■

5 CONCLUSION

We have examined the implications of the intolerance of a large decline in wealth on optimal consumption and portfolio policies for an investor with constant relative risk aversion preferences. Our specification of preferences is an attempt to capture two features we think are important for understanding investor behavior. The first one is related to the concept of loss aversion over wealth. Essentially, a heavy loss in the stock market may inflict a severe blow to the ego of an investor who comes to realize that she is not the gifted investor she once thought to be. She may experience some regret about her investment and even a feeling of humiliation (being a loser) as the news spread among friends and family members. The second motivation reflects the fact that when an investor's wealth is increasing, she may psychologically commit part of the profits made for future expenditures, including sometimes important lifetime decisions such as early retirement for instance. Being forced to revise downward her plans after a drop in wealth may be painful.

We find that wealth ratcheting induces a ratcheting behavior of consumption as current optimal consumption is always maintained above a fixed percentage of its all-time maximum. Hedging motives and mitigating the cost associated with the ratchet feature of the constraint govern the agent's intertemporal choices. We have isolated the impact of hedging by analyzing asset management for a university that is required to preserve its endowment. Essentially, the lifetime utility relative risk aversion rises, which leads to smaller stock holdings. Looking at the wedge between optimal allocations for the fixed floor and a ratchet floor allows us to quantify the ratchet impact and uncover its significance. In particular, impatient investors may curb investment in stocks as wealth approaches its peak to limit its growth and the risk of raising the minimum floor. In fact, for large values of the drawdown coefficient, investors whose pure time discount rate exceeds the interest rate find it optimal not to let the wealth process achieve a new maximum. An extension of the basic model incorporates the spirit of capitalism and interprets wealth as an index for social status. Lasting benefits from current and future status levels provide incentives for a higher growth of the wealth and induce a more aggressive risky investment strategy at the expense of consumption.

Possible extension of this paper would be to include labor income. If the correlation between labor income and the stock market is small or negative, the investor naturally would like to borrow against her future income to increase risky investment, which could drive down her financial wealth and exacerbate both hedging motives and the ratchet effect. A detailed analysis is left for future research.

6 APPENDIX

APPENDIX A

Proof of Lemma 1. Consider the following changes of variables: $X = F'(W), W = -J'(X)$ and $F(W) = J(X) - XJ'(X)$. Using relationship (11), we find that the function J must be solution of the following linear ODE

$$\theta J(X) = \frac{bX^{\frac{b-1}{b}}}{1-b} + (\theta - r)XJ'(X) + \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 X^2 J''(X).$$

The general solution of this ODE is

$$J(X) = \frac{bAX^{\frac{b-1}{b}}}{1-b} - \frac{bL_1}{\beta_1 - 1 + b}X^{\frac{\beta_1 - 1 + b}{b}} - \frac{bL_2}{\beta_2 - 1 + b}X^{\frac{\beta_2 - 1 + b}{b}}, \tag{18}$$

where L_1 and L_2 are constants. Differentiating (18) with respect to X and using the fact that $X = F'(W)$ and $W = -J'(X)$ provides the desired result. ■

Proof of Properties $\beta_1 > 1$ and $1 - b - \beta_2 > 0$. Recall that β_1 is the positive root of the quadratic

$$Q(x) = \frac{1}{2}\left(\frac{\mu - r}{b\sigma}\right)^2 x^2 + \left(\frac{1}{A} - r - \frac{1}{2}\left(\frac{\mu - r}{b\sigma}\right)^2\right)x - \frac{1}{A}.$$

Since $Q(1) = -r < 0$, we must have $\beta_1 > 1$. Then, using the fact that $\frac{1}{2}\left(\frac{\mu-r}{b\sigma}\right)^2 A\beta_2\beta_1 = 1$ and $\frac{\beta_1+\beta_2}{2}\left(\frac{\mu-r}{b\sigma}\right)^2 = -\frac{1}{A} + r + \frac{1}{2}\left(\frac{\mu-r}{b\sigma}\right)^2$, we find that $(\beta_1 + b - 1)(1 - b - \beta_2) = \frac{\theta}{\frac{1}{2}\left(\frac{\mu-r}{b\sigma}\right)^2}$. Since $\beta_1 > 1$, indeed we have $1 - b - \beta_2 > 0$. ■

APPENDIX B

Proof of Properties P1. F is strictly increasing in W and decreasing in M since given W, the higher M, the more stringent the drawdown constraint. Let $\lambda \in (0,1)$, (W_0, M_0) and (W_0', M_0') be two initial states and (c, z) and (c', z') the associated optimal strategies. Then, for initial wealth level $\lambda W_0 + (1-\lambda)W_0'$ and initial all-time high wealth level $\lambda M_0 + (1-\lambda)M_0'$, $(\lambda c + (1-\lambda)c', \lambda z + (1-\lambda)z')$ is also a feasible strategy as the wealth dynamics are linear in variables (c, z) and

$$\begin{aligned}\lambda W_t + (1-\lambda)W_t' &\geq \lambda\alpha M_t + (1-\lambda)\alpha M_t' \\ &\geq \alpha \max\{\lambda M_0 + (1-\lambda)M_0', \lambda W_s + (1-\lambda)W_s', s \leq t\}.\end{aligned}$$

Finally, by strict concavity of the utility function u

$$E_0\left[\int_0^\infty u(\lambda c_s + (1-\lambda)c_s')e^{-\theta s}ds\right] > E_0\left[\int_0^\infty \left(\lambda u(c_s) + (1-\lambda)u(c_s')\right)e^{-\theta s}ds\right],$$

which implies that $F(\lambda W_0 + (1-\lambda)W_0', \lambda M_0 + (1-\lambda)M_0') > \lambda F(W_0, M_0) + (1-\lambda)F(W_0', M_0')$. ■

Proof of Properties P2. Let (c, z) be a feasible strategy for an initial state (W_0, M_0). Then for $\lambda > 0$, $(\lambda c, \lambda z)$ is feasible for the initial state $(\lambda W_0, \lambda M_0)$ since the dynamics of the corresponding wealth process W_λ are

$$\begin{aligned}dW_{\lambda s} &= (\lambda r W_s - \lambda c_s + \lambda z_s(\mu - r))ds + \lambda z_s \sigma dw_s \\ &= \lambda dW_s,\end{aligned}$$

so $W_{\lambda s} = \lambda W_s$ and therefore $W_{\lambda s} = \lambda W_s \geq \alpha\lambda M_s = \alpha M_{\lambda s}$. It follows that $F(\lambda W, \lambda M) \geq \lambda^{1-b} F(W, M)$ by homogeneity of degree $1-b$ the utility function. Then, observe that $F(W, M) = F(\lambda^{-1}\lambda W, \lambda^{-1}\lambda M) \geq \lambda^{b-1} F(\lambda W, \lambda M)$, so in fact we have $F(\lambda W, \lambda M) = \lambda^{1-b} F(W, M)$. ■

APPENDIX C

Derivation of Boundary Conditions

Condition for a Well Defined Value Function. The boundary conditions must be such that f is well defined and the drawdown constraint is satisfied. Differentiating relationship (12) with respect to u yields

$$-\frac{bf'(u)}{f''(u)} = A(f'(u))^{-\frac{1}{b}} - (\beta_1 - 1)K_1(f'(u))^{\frac{\beta_1 - 1}{b}} - (\beta_2 - 1)K_2(f'(u))^{\frac{\beta_2 - 1}{b}}. \tag{19}$$

Since $-\frac{bf'(u)}{f''(u)}$ is non-negative, for all u in $[\alpha, 1]$, we must have

$$(\beta_1 - 1)K_1(f'(u))^{\frac{\beta_1}{b}} + (\beta_2 - 1)K_2(f'(u))^{\frac{\beta_2}{b}} \leq A. \tag{20}$$

Then, set $Y = (f'(1))^{\frac{1}{b}}$ and $X = (f'(\alpha))^{\frac{1}{b}}$ and note that $Y \leq X$. Given relationship (12), it must be the case that for all y in $[Y, X]$, the function $G : y \mapsto Ay^{-1} + K_1 y^{\beta_1 - 1} + K_2 y^{\beta_2 - 1}$ is invertible so we can write $f'(u) = \left(G^{-1}(u)\right)^b$, for all u in $[\alpha, 1]$. Since f'' is negative, then G' must be negative. Condition (20) is equivalent to $\Gamma(y) = -A + (\beta_1 - 1)K_1 y^{\beta_1} + (\beta_2 - 1)K_2 y^{\beta_2} < 0$. As shown in the sequel, we must have $\Gamma(X) = 0$. Since $\Gamma'(y) = \beta_1(\beta_1 - 1)K_1 y^{\beta_1 - 1} + \beta_2(\beta_2 - 1)K_2 y^{\beta_2 - 1}$, for $K_1 > 0$ and $K_2 < 0$ (to be justified in the sequel), the function Γ' is strictly increasing and has at most one root on $[Y, X]$. Hence, $\Gamma(Y) \leq 0$ is a necessary and sufficient condition to guarantee that $\Gamma < 0$ on (Y, X). ■

Boundary Condition at $u = 1$. First of all, from relationship (12) we have

$$1 = A(f'(1))^{-\frac{1}{b}} + K_1(f'(1))^{\frac{\beta_1 - 1}{b}} + K_2(f'(1))^{\frac{\beta_2 - 1}{b}}. \tag{21}$$

Then, for $h > 0$, over the interval of time $[t, t+h]$, the HJB equation is

$$F(W_t, M_t) = \max_{(c_t, z_t)} E_t\left[u(c_t)h + e^{-\theta h}F(W_{t+h}, M_{t+h})\right],$$

so using Ito lemma for semi-martingales

$$\begin{aligned} 0 &= \max_{(c_t,z_t)} u(c_t)h + E_t\left[\int_t^{t+h}\left(-\theta F + (rW_s - c_s + z_s(\mu - r))F_1 + \frac{\sigma^2}{2}z_s^2 F_{11}\right)ds\right] \\ &\quad + E_t\left[\int_t^{t+h} F_2 dM_s\right] \end{aligned} \tag{22}$$

As derived in Grossman and Zhou [21]

$$E_t\left[M_{t+h} - M_t \mid W_t = M_t\right] = \sqrt{\frac{2}{\pi}}\sigma\left|z_t\right|\sqrt{h} + O(h).$$

When h is small, $\sqrt{h}$ dominates h so in order for the Bellman equation to hold at $W = M$, we must have $F_2(M, M) = 0$ or $z^*(1) = 0$. Since HJB involves a maximization over z, *whenever feasible*, it is optimal to choose $z^*(1) = -\frac{(\mu - r)f'(1)}{\sigma^2 f''(1)}$, instead of $z^*(1) = 0$. It remains to investigate the impact of the term of order h from the maximum wealth process on the HJB equation. Recall that $F_2(M,M) \leq 0$. When $z^*(1) = 0$ is optimal and $F_2(M,M) < 0$, suppose that we choose a strategy at $W = M$ for the consumption process such that $dM_s > 0$, thus we have $F_2 dM_s < 0$. This implies that the maximum value achievable for the HJB in relationship (22) must be strictly lower than in the case of a strategy that yields $dM_s = 0$. Proposition 1. of the paper establishes that there exists a strategy at $W = M$ for the consumption that indeed achieves $dM_s = 0$ and therefore it must be the optimal one. Thus, for the optimal strategy $dM_s = 0$. We conclude that in *both* cases $F_2 dM_s = 0$, and the expression for (reduced) HJB given by relationship (12) is also valid at $u = 1$.

Case 1: $F_2(M,M) = 0$**.** This is equivalently to $f'(1) = (1-b)f(1)$. Using relationship (11) leads to

$$\begin{aligned} \frac{\theta}{1-b} &= \left(\frac{b}{1-b} + (r + \frac{1}{2b}\left(\frac{\mu - r}{\sigma}\right)^2)A\right)(f'(1))^{-\frac{1}{b}} + \left(r - \frac{\beta_1 - 1}{2b}\left(\frac{\mu - r}{\sigma}\right)^2\right)K_1(f'(1))^{\frac{\beta_1 - 1}{b}} \\ &+ \left(r - \frac{\beta_2 - 1}{2b}\left(\frac{\mu - r}{\sigma}\right)^2\right)K_2(f'(1))^{\frac{\beta_2 - 1}{b}}. \end{aligned}$$

Using the definition of A, relationship (21) and the fact that β_1 and β_2 are the roots of the quadratic (6), it follows that

$$-\beta_1\beta_2\left(A(f'(1))^{-\frac{1}{b}} - 1\right) = (1-b)\left(\beta_1 K_1(f'(1))^{\frac{\beta_1 - 1}{b}} + \beta_2 K_2(f'(1))^{\frac{\beta_2 - 1}{b}}\right). \blacksquare$$

Case 2: $z^*(1) = 0$**.** Imposing $z^*(1) = 0$ leads to

$$A = (\beta_1 - 1)K_1(f'(1))^{\frac{\beta_1}{b}} + (\beta_2 - 1)K_2(f'(1))^{\frac{\beta_2}{b}}. \blacksquare$$

Boundary Condition at $u = \alpha$**.** At $W = \alpha M$, as in the fixed floor problem, risky investment must be zero, so from relationship (15), we must have

$$A = (\beta_1 - 1)K_1(f'(\alpha))^{\frac{\beta_1}{b}} + (\beta_2 - 1)K_2(f'(\alpha))^{\frac{\beta_2}{b}}.$$

Then, at $u = \alpha$, relationship (12) provides the following condition

$$\alpha = A(f'(\alpha))^{-\frac{1}{b}} + K_1(f'(\alpha))^{\frac{\beta_1 - 1}{b}} + K_2(f'(\alpha))^{\frac{\beta_2 - 1}{b}}.$$

To summarize, the boundary conditions are:

$$\begin{aligned} &\alpha X = A + K_1 X^{\beta_1} + K_2 X^{\beta_2} \\ &A = (\beta_1 - 1)K_1 X^{\beta_1} + (\beta_2 - 1)K_2 X^{\beta_2} \\ &Y = A + K_1 Y^{\beta_1} + K_2 Y^{\beta_2} \\ &\begin{cases} \beta_1\beta_2\,(Y - A) = (1-b)\left(\beta_1 K_1 Y^{\beta_1} + \beta_2 K_2 Y^{\beta_2}\right) \text{ if } F_2(M,M) = 0 \\ A = (\beta_1 - 1)K_1 Y^{\beta_1} + (\beta_2 - 1)K_2 Y^{\beta_2} \text{ if } z^*(1) = 0. \end{cases} \end{aligned}$$

since $A_0 = -\frac{\beta_1\beta_2 A}{1-b-\beta_1\beta_2}$, the system can be rewritten as stipulated in the core of the paper. ■

APPENDIX D

APPENDIX D1

Proof of Existence and Uniqueness of (X, Y, K_1, K_2) **when** $F_2(M, M) = 0$ is imposed. We want to show existence and uniqueness for system S

$$\begin{aligned} \alpha X &= A + K_1 X^{\beta_1} + K_2 X^{\beta_2} & (23) \\ A &= (\beta_1 - 1) K_1 X^{\beta_1} + (\beta_2 - 1) K_2 X^{\beta_2} & (24) \\ Y &= A + K_1 Y^{\beta_1} + K_2 Y^{\beta_2} & (25) \\ \beta_1 \beta_2 (Y - A) &= (1 - b) \left(\beta_1 K_1 Y^{\beta_1} + \beta_2 K_2 Y^{\beta_2} \right). & (26) \end{aligned}$$

Combining relationships (25) and (26) leads to $\beta_1(1 - b - \beta_2) K_1 Y^{\beta_1} + \beta_2(1 - b - \beta_1) K_2 Y^{\beta_2} = 0$. Since both $\beta_1(1 - b - \beta_2)$ and $\beta_2(1 - b - \beta_1)$ are positive, it must be the case that K_1 and K_2 have opposite signs. Then

$$K_1 Y^{\beta_1} = \frac{\beta_2(1 - b - \beta_1)}{\beta_1 - \beta_2} \frac{Y - A}{b - 1},$$

which implies that K_1 has the same sign as $\frac{Y-A}{b-1}$. Eliminating K_1 and K_2 from relationship (24) yields

$$A = \left(\beta_2(\beta_1 - 1)(1 - b - \beta_1) \left(\frac{X}{Y}\right)^{\beta_1} - \beta_1(\beta_2 - 1)(1 - b - \beta_2) \left(\frac{X}{Y}\right)^{\beta_2} \right) \frac{Y - A}{(b - 1)(\beta_1 - \beta_2)}.$$

Since $\beta_2(\beta_1 - 1)(1 - b - \beta_1) > 0$ and $-\beta_1(\beta_2 - 1)(1 - b - \beta_2) > 0$, we find that $\frac{Y-A}{b-1} > 0$. Hence $K_1 > 0$ **and** $K_2 < 0$ and $Y \geq A$ ($Y \leq A$) exactly when $b \geq 1$ ($b \leq 1$). Combining (23) and (24) leads to $\alpha X = \beta_1 K_1 X^{\beta_1} + \beta_2 K_2 X^{\beta_2}$, and eliminating K_1 and K_2 using relationships (25) and (26) yields

$$\alpha X = \beta_1 \beta_2 \left((1 - b - \beta_1) \left(\frac{X}{Y}\right)^{\beta_1} - (1 - b - \beta_2) \left(\frac{X}{Y}\right)^{\beta_2} \right) \frac{Y - A}{(b - 1)(\beta_1 - \beta_2)}.$$

Set $\varpi = \frac{X}{Y} \geq 1$, we have

$$X = \frac{\beta_1 \beta_2 A \left((1 - b - \beta_1) \varpi^{\beta_1} - (1 - b - \beta_2) \varpi^{\beta_2} \right)}{\alpha \left(\beta_2(\beta_1 - 1)(1 - b - \beta_1) \varpi^{\beta_1} - \beta_1(\beta_2 - 1)(1 - b - \beta_2) \varpi^{\beta_2} \right)}, \tag{27}$$

and ϖ is implicitly defined by $\Phi_\alpha(\varpi) = 0$ where for $x \geq 1$, the auxiliary function Φ_α is defined by

$$\begin{aligned} \Phi_\alpha : \quad x \quad \mapsto \quad & \alpha \left(\beta_2(\beta_1 - 1)(1 - b - \beta_1) x^{\beta_1} - \beta_1(\beta_2 - 1)(1 - b - \beta_2) x^{\beta_2} \right) \\ & -\beta_1 \beta_2 \left((1 - b - \beta_1) x^{\beta_1 - 1} - (1 - b - \beta_2) x^{\beta_2 - 1} \right) - \alpha(1 - b)(\beta_1 - \beta_2). \end{aligned}$$

We want to show that for $\alpha < 1$, Φ_α has a unique root $\varpi > 1$. Φ_α is continuously differentiable and $\Phi_\alpha(1) = (1 - \alpha)\beta_1 \beta_2(\beta_1 - \beta_2) < 0$. Then, we show that $\Phi_\alpha(\frac{1}{\alpha}) < 0$. A little bit of algebra yields

$$\Phi_\alpha(\frac{1}{\alpha}) = -\beta_2(1 - b - \beta_1) \left(\frac{1}{\alpha}\right)^{\beta_1 - 1} + \beta_1(1 - b - \beta_2) \left(\frac{1}{\alpha}\right)^{\beta_2 - 1} - \alpha(1 - b)(\beta_1 - \beta_2).$$

For $x > 1$, define an auxiliary function

$$\Theta : \quad x \quad \mapsto \quad -\beta_2(1 - b - \beta_1) x^{\beta_1} + \beta_1(1 - b - \beta_2) x^{\beta_2}.$$

Again Θ is continuous and differentiable and $\lim\limits_{1} \Theta = (1 - b)(\beta_1 - \beta_2)$. Clearly, Θ is decreasing, which implies that for all x in $(1, \infty)$, $\Theta(x) < (1 - b)(\beta_1 - \beta_2)$ and in particular $\Phi_\alpha(\frac{1}{\alpha}) < 0$. Then $\Phi'_\alpha(x) = -\beta_1 \beta_2 x^{\beta_2 - 2}(\alpha x - 1)\Psi(x)$, where $\Psi(x) = -(\beta_1 - 1)(1 - b - \beta_1) x^{\beta_1 - \beta_2} + (\beta_2 - 1)(1 - b - \beta_2)$.

Since $-(\beta_1-1)(1-b-\beta_1)>0$ and $\beta_1-\beta_2>0$, Ψ is strictly increasing and $\Psi(1)=-(\beta_1-1)(1-b-\beta_1)+(\beta_2-1)(1-b-\beta_2)$. Note that

$$\begin{aligned}\Psi(1)\geq 0 \quad &\Leftrightarrow \quad \tfrac{-\beta_1\beta_2 A}{1-b-\beta_1\beta_2}\leq \tfrac{\beta_1\beta_2 A}{(\beta_1-1)(\beta_2-1)}\\ &\Leftrightarrow \quad A_0\leq \tfrac{1}{r}\\ &\Leftrightarrow \quad r<\theta.\end{aligned}$$

Case 1: $\Psi(1)\geq 0$. $\Psi>0$ and therefore Φ_α is decreasing on $\left[1,\frac{1}{\alpha}\right]$ and increasing on $[\frac{1}{\alpha},\infty)$. Since $\lim_\infty \Phi_\alpha=\infty$ we conclude that Φ_α has a unique root $\varpi\in[\frac{1}{\alpha},\infty)$. Note that $\Psi(\varpi)>0$. ■

Case 2: $\Psi(1)<0$. Then define x^* such that $\Psi(x^*)=0$, i.e.

$$x^*=\left(\frac{(\beta_2-1)(1-b-\beta_2)}{(\beta_1-1)(1-b-\beta_1)}\right)^{\frac{1}{\beta_1-\beta_2}}.$$

It follows that $\Psi<0$ on $[1,x^*]$ and $\Psi>0$ on $[x^*,\infty)$.

Case 2.1: $\frac{1}{\alpha}<x^*$. Φ_α is increasing on $\left[1,\frac{1}{\alpha}\right]$, decreasing on $\left[\frac{1}{\alpha},x^*\right]$ and increasing on $[x^*,\infty)$. Since $\Phi_\alpha(\frac{1}{\alpha})<0$, we conclude that Φ_α has a unique root that belongs to $[x^*,\infty)$. Again, note that $\Psi(\varpi)>0$. ■

Case 2.2: $x^*<\frac{1}{\alpha}$. Φ_α is increasing on $[1,x^*]$, then decreasing on $\left[x^*,\frac{1}{\alpha}\right]$ and increasing on $[\frac{1}{\alpha},\infty)$. It remains to show that $\Phi_\alpha(x^*)<0$ to conclude that Φ_α has a unique root in $[\frac{1}{\alpha},\infty)$. Using the definition of x^*, one can show that

$$\Phi_\alpha(x^*)=(\beta_1-\beta_2)(\beta_1+b-1)\left(\alpha(\beta_1-1)(x^*)^{\beta_1}-\frac{\beta_1\beta_2}{\beta_2-1}(x^*)^{\beta_1-1}\right)-\alpha(1-b)(\beta_1-\beta_2).$$

For $x>1$, define an auxiliary function

$$\Xi:\quad x\quad\mapsto\quad \alpha(\beta_1-1)x^{\beta_1}-\tfrac{\beta_1\beta_2}{\beta_2-1}x^{\beta_1-1}.$$

Ξ is continuous and differentiable and $\Xi'(x)=\beta_1(\beta_1-1)x^{\beta_1-2}(\alpha x-\frac{\beta_2}{\beta_2-1})$. It follows that Ξ is decreasing on $\left[1,\frac{\beta_2}{\alpha(\beta_2-1)}\right]$ and increasing on $\left[\frac{\beta_2}{\alpha(\beta_2-1)},\frac{1}{\alpha}\right]$. Since $(\beta_1-\beta_2)(\beta_1+b-1)>0$, it must be the case that $\Phi_\alpha(x^*)\leq\max\{\Phi_\alpha(1),\Phi_\alpha(\frac{1}{\alpha})\}$, so in particular $\Phi_\alpha(x^*)<0$ and the desired result follows. Again, note that $\Psi(\varpi)>0$.

To summarize, there is a unique real number $\varpi>\frac{1}{\alpha}$ such that $\Phi_\alpha(\varpi)=0$. In addition, we have $\Phi'_\alpha(\varpi)>0$. From the definition of ϖ, we have

$$\frac{1}{\alpha}\Phi'_\alpha(\varpi)\frac{\partial\varpi}{\partial\alpha}=-\frac{\beta_1\beta_2\varpi^{\beta_2-1}}{\alpha^2}\left((1-b-\beta_1)\varpi^{\beta_1-\beta_2}-(1-b-\beta_2)\right).$$

Since $\varpi>1$, we have $(1-b-\beta_1)\varpi^{\beta_1-\beta_2}-(1-b-\beta_2)<-(\beta_1-\beta_2)<0$, we obtain that $\frac{\partial\varpi}{\partial\alpha}<0$. Setting $\alpha_c=\frac{Y}{X}$, this implies that $\frac{\partial\alpha_c}{\partial\alpha}<0$. The existence and uniqueness of X,Y,K_1 and K_2 follow. When $b=1$, $Y=A$ and ϖ is defined by

$$\varpi^{\beta_1}(\beta_1-\alpha(\beta_1-1)\varpi)=\varpi^{\beta_2}(\beta_2-\alpha(\beta_2-1)\varpi). \tag{28}$$

Finally, we check that $r-\frac{1}{\alpha X}>0$. Using relationship (27), elementary algebra shows that condition $r-\frac{1}{\alpha X}>0$ is equivalent to $-(\beta_1-1)(1-b-\beta_1)\varpi^{\beta_1}+(\beta_2-1)(1-b-\beta_2)\varpi^{\beta_1}>1$. This condition is the same as $\Psi(\varpi) > 0$, which is satisfied. ■

APPENDIX D2

Proof of Existence and Uniqueness of (X, Y, K_1, K_2) **when** $z^*(1) = 0$ is imposed. We want to show existence and uniqueness for system S'

$$\begin{aligned} \alpha X &= A + K_1 X^{\beta_1} + K_2 X^{\beta_2} && (29)\\ A &= (\beta_1 - 1)K_1 X^{\beta_1} + (\beta_2 - 1)K_2 X^{\beta_2} && (30)\\ Y &= A + K_1 Y^{\beta_1} + K_2 Y^{\beta_2} && (31)\\ A &= (\beta_1 - 1)K_1 Y^{\beta_1} + (\beta_2 - 1)K_2 Y^{\beta_2}. && (32) \end{aligned}$$

Note that K_1 and K_2 must have opposite sign otherwise the function

$$\Lambda : y \mapsto (\beta_1 - 1)K_1 y^{\beta_1} + (\beta_2 - 1)K_2 y^{\beta_2},$$

is monotonic and therefore the equation $\Lambda(y) = A$ cannot have two distinct roots. Then we have $\alpha X = \beta_1 K_1 X^{\beta_1} + \beta_2 K_2 X^{\beta_2}$, and since $X > 0, \beta_1 > 0$ and $\beta_2 < 0$, it must be the case that $K_1 > 0$ and $K_2 < 0$. Then combining relationships (31) and (32) yields

$$\begin{aligned} (\beta_1 - \beta_2)K_1 Y^{\beta_1} &= \beta_2 A - (\beta_2 - 1)Y \\ -(\beta_1 - \beta_2)K_2 Y^{\beta_2} &= \beta_1 A - (\beta_1 - 1)Y. \end{aligned}$$

Once again, define $\varpi = \frac{X}{Y} \geq 1$. Eliminating K_1 and K_2 leads to

$$\begin{aligned} (\beta_1 - 1)(\beta_2 A - (\beta_2 - 1)Y)\varpi^{\beta_1} - (\beta_2 - 1)(\beta_1 A - (\beta_1 - 1)Y)\varpi^{\beta_2} &= (\beta_1 - \beta_2)A \\ (\beta_2 A - (\beta_2 - 1)Y)\varpi^{\beta_1} - (\beta_1 A - (\beta_1 - 1)Y)\varpi^{\beta_2} &= (\beta_1 - \beta_2)(\alpha X - A). \end{aligned}$$

Eliminating Y leads to

$$X = \frac{A\left((\beta_1 - \beta_2)\varpi^{\beta_1+\beta_2} + \beta_1(\beta_2 - 1)\varpi^{\beta_1} - \beta_2(\beta_1 - 1)\varpi^{\beta_2}\right)}{\alpha(\beta_1 - 1)(\beta_2 - 1)(\varpi^{\beta_1} - \varpi^{\beta_2})}, \tag{33}$$

and ϖ is implicitly defined by $\Phi_\alpha(\varpi) = 0$ where for $x \geq 1$, the auxiliary function Φ_α is defined by

$$\Phi_\alpha : \quad x \quad \mapsto \quad \begin{aligned} &-\alpha\beta_2(\beta_1 - 1)x^{\beta_1} + \beta_1(\beta_2 - 1)x^{\beta_1 - 1} + (\beta_1 - \beta_2)x^{\beta_1+\beta_2-1} \\ &-\beta_2(\beta_1 - 1)x^{\beta_2 - 1} + \alpha\beta_1(\beta_2 - 1)x^{\beta_2} + \alpha(\beta_1 - \beta_2). \end{aligned}$$

We want to show that for $\alpha < 1$, Φ_α has a unique root $\varpi > 1$. Φ_α is continuously differentiable and $\Phi_\alpha(1) = 0$. Then

$$\begin{aligned} \Phi'_\alpha(x) \quad = \quad & -\alpha\beta_1\beta_2(\beta_1 - 1)x^{\beta_1 - 1} + \beta_1(\beta_2 - 1)(\beta_1 - 1)x^{\beta_1 - 2} + (\beta_1 - \beta_2)\beta_1 + \beta_2 - 1)x^{\beta_1+\beta_2-2} \\ & -\beta_2(\beta_1 - 1)(\beta_2 - 1)x^{\beta_2 - 2} + \alpha\beta_1\beta_2(\beta_2 - 1)x^{\beta_2 - 1}. \end{aligned}$$

$\Phi'_\alpha(1) = -\beta_1\beta_2(\beta_1 - \beta_2)(\alpha - 1) < 0$. In addition, for all $x \geq 1$, $\Phi'_\alpha(x)$ has the same sign as $\Theta(x)$ where

$$\begin{aligned} &\Theta(x) = -\alpha\beta_1\beta_2(\beta_1 - 1)x^{1-\beta_2} + \beta_1(\beta_2 - 1)(\beta_1 - 1)x^{-\beta_2} + (\beta_1 - \beta_2)\beta_1 + \beta_2 - 1) \\ &-\beta_2(\beta_1 - 1)(\beta_2 - 1)x^{-\beta_1} + \alpha\beta_1\beta_2(\beta_2 - 1)x^{1-\beta_1}. \end{aligned}$$

We have $\Theta(1) = -\beta_1\beta_2(\beta_1 - \beta_2)(\alpha - 1) < 0$ and

$$\Theta'(x) = \alpha\beta_1\beta_2(\beta_1 - 1)(\beta_2 - 1)\,(\alpha x - 1)\,x^{-\beta_1-1}(x^{\beta_1-\beta_2} - 1).$$

Hence, the function Θ is strictly decreasing on $[1, \frac{1}{\alpha}]$ and strictly increasing on $[\frac{1}{\alpha}, \infty)$. Since $\lim_{\infty} \Theta = \infty$, we conclude that Θ has a unique root y^* in $(\frac{1}{\alpha}, \infty)$. Thus, Φ_α is decreasing on $[1, y^*]$ and increasing on $[y^*, \infty)$ with $\lim_{\infty} \Phi_\alpha = \infty$. This shows that Φ_α has a unique root $\varpi > \frac{1}{\alpha}$ and $\Phi'_\alpha(\varpi) > 0$. From the definition of ϖ, we have

$$\Phi'_\alpha(\varpi)\frac{\partial \varpi}{\partial \alpha} = \beta_2(\beta_1 - 1)\varpi^{\beta_1} - \beta_1(\beta_2 - 1)\varpi^{\beta_2} - (\beta_1 - \beta_2).$$

Since $x \mapsto \beta_2(\beta_1-1)x^{\beta_1} - \beta_1(\beta_2-1)x^{\beta_2} - (\beta_1-\beta_2)$ is decreasing and $\varpi > 1$, we have $\beta_2(\beta_1-1)\varpi^{\beta_1} - \beta_1(\beta_2-1)\varpi^{\beta_2} - (\beta_1-\beta_2) < 0$. Hence $\frac{\partial \varpi}{\partial \alpha} < 0$. Setting $\alpha_c = \frac{Y}{X}$, this implies that $\frac{\partial \alpha_c}{\partial \alpha} < 0$. The existence and uniqueness of X, Y, K_1 and K_2 follow. Finally, we check that $r - \frac{1}{\alpha X} > 0$. Using relationship (33), elementary algebra shows that condition $r - \frac{1}{\alpha X} > 0$ is equivalent to

$$(\beta_1 - \beta_2)\varpi^{\beta_1+\beta_2} - \beta_1\varpi^{\beta_1} + \beta_2\varpi^{\beta_2} > 0. \tag{34}$$

For $x \geq 1$, define the auxiliary function

$$\Xi : x \mapsto (\beta_1 - \beta_2)x^{\beta_1} - \beta_1 x^{\beta_1-\beta_2} + \beta_2.$$

Condition (34) is equivalent to $\Xi(\varpi) > 0$. Then $\Xi'(x) = \beta_1(\beta_1 - \beta_2)x^{\beta_1-1}(1 - x^{-\beta_2}) > 0$, for $x > 1$ since $\beta_2 < 0$. Ξ is an increasing function with $\Xi(1) = 0$, so we must have $\Xi(\varpi) > 0$. ■

APPENDIX E

Proof of Proposition 1

Step 0: The value function F to be well defined iff

$$(\beta_1 - 1)K_1Y^{\beta_1} + (\beta_2 - 1)K_2Y^{\beta_2} \leq A. \tag{35}$$

If $F_2(M, M) = 0$ is imposed, since

$$A - (\beta_1 - 1)K_1Y^{\beta_1} - (\beta_2 - 1)K_2Y^{\beta_2} = \frac{1 - b - \beta_1\beta_2}{b - 1}(A_0 - Y),$$

we must have $Y \leq A_0$ ($Y \geq A_0$) if $b \geq 1 (b \leq 1)$. Conversely, if the solution of system S is such that $Y \leq A_0$ ($Y \geq A_0$) whenever $b \geq 1 (b \leq 1)$, then condition (35) is satisfied, and the value function F is well defined; therefore $F_2(M, M) = 0$ is optimal. ■

Step 1: Around $\alpha = 0$, $F_2(M, M) = 0$ is always optimal.

The solution K_1 of system S is such that $K_1 > 0$ and $\frac{\partial K_1}{\partial \alpha} > 0$ (see appendix F), so K_1 must have a finite (non-negative) limit when α goes to 0. Since $K_1Y^{\beta_1} = \frac{\beta_2(1-b-\beta_1)}{\beta_1-\beta_2}\frac{Y-A}{b-1}$, we can conclude that Y also has a finite non-negative limit when α goes to 0. This implies that K_2 also a finite non-positive limit when α goes to 0. Then recall that $(\beta_1 - \beta_2)K_2Y^{\beta_2} = (\beta_1 - 1)\alpha X - \beta_1 A < 0$, and $(\beta_1 - \beta_2)K_1Y^{\beta_1} = -(\beta_2 - 1)\alpha X + \beta_2 A > 0$, so $\frac{\beta_2 A}{\beta_2-1} \leq \alpha X \leq \frac{\beta_1 A}{\beta_1-1}$. Finally, writing

$$(\beta_1 - 1)(\alpha X - \frac{\beta_1 A}{\beta_1 - 1}) = (\beta_1 - \beta_2)K_2(\alpha X)^{\beta_2}\alpha^{-\beta_2}.$$

Since $\beta_2 < 0$ and αX is bounded, we have $\lim_{\alpha\to 0}(\alpha X)^{\beta_2}\alpha^{-\beta_2} = 0$, which implies that $\lim_{\alpha\to 0}\alpha X = \frac{\beta_1 A}{\beta_1-1}$. It follows easily that $\lim_{\alpha\to 0} K_2Y^{\beta_2} = \lim_{\alpha\to 0} K_1Y^{\beta_1} = 0$, and therefore $\lim_{\alpha\to 0} Y = A$. Since for $b > 1 (b < 1)$,

$A \leq A_0 (A \geq A_0)$, we can conclude that the condition of proposition 1. is satisfied so around $\alpha = 0$, $F_2(M, M) = 0$ is optimal. ■

Step 2: Behavior of systems S and S' around $\alpha = 1$.

$$\text{For system } S, \lim_{\alpha \to 1} Y = \begin{cases} A_0, \text{ if } r \geq \theta \\ > A_0 (< A_0), \text{ when } b > 1 (b < 1), \text{ if } r < \theta \end{cases}$$
$$\text{For system } S', \lim_{\alpha \to 1} Y = \frac{1}{r}$$

Proof: System S. For $0 < \alpha < 1$, from appendix D1. the equation $\Phi_\alpha(x) = 0$ for $x \geq 1$ has a unique solution $\varpi = \frac{X}{Y} > 1$. When $\alpha = 1$, $\Phi_1(x) = 0$ has at most two solutions, including value 1. We want to show that $\lim_{\alpha \to 1} \varpi = 1$ and $\lim_{\alpha \to 1} Y = A_0$. Set $k = \frac{(\beta_1 - 1)(1 - b - \beta_1)}{(\beta_2 - 1)(1 - b - \beta_2)} > 0$. It is easy to verify that $k \geq 1 (\leq 1)$ iff $r \geq \theta (\leq \theta)$. Then, for $x \geq 1$, recall that

$$\begin{aligned} \Phi_1(x) &= \beta_2(\beta_1 - 1)(1 - b - \beta_1)x^{\beta_1} - \beta_1(\beta_2 - 1)(1 - b - \beta_2)x^{\beta_2} \\ &\quad -\beta_1\beta_2\left((1 - b - \beta_1)x^{\beta_1 - 1} - (1 - b - \beta_2)x^{\beta_2 - 1}\right) + (b - 1)(\beta_1 - \beta_2). \end{aligned}$$

Φ_1 is a smooth function with $\Phi_1(1) = 1$ and

$$\Phi_1'(x) = \frac{\beta_1\beta_2}{x}\left((\beta_1 - 1)(1 - b - \beta_1)(x^{\beta_1} - x^{\beta_1 - 1}) - (\beta_2 - 1)(1 - b - \beta_2)(x^{\beta_2} - x^{\beta_2 - 1})\right).$$

Again, we have $\Phi_1'(1) = 0$ and note that $\Phi_1'(x)$ has the same sign as $\Gamma(x) = k(x^{\beta_1 - \beta_2 - 1} - x^{\beta_1 - \beta_2 - 2}) - (x - 1)$. Γ is a smooth function with $\Gamma(1) = 0$ and

$$\Gamma'(x) = k((\beta_1 - \beta_2 - 1)x^{\beta_1 - \beta_2 - 2} - (\beta_1 - \beta_2 - 2)x^{\beta_1 - \beta_2 - 3}) - 1.$$

$\Gamma'(1) = k - 1$ and it is easy to see that for $k > 1$, $\Gamma''(x) > 0$ for all $x > 1$.

Case 1: $r \geq \theta$. It follows that $\Gamma' \geq 0$, so $\Phi_1' \geq 0$ and for $x > 1$, $\Phi_1(x) > 0$. This implies that $x = 1$ is the unique solution of $\Phi_1(x) = 0$, and thus $\lim_{\alpha \to 1} X = \lim_{\alpha \to 1} Y$. If $\alpha = 1$, System S becomes a 3 by 3 system

$$\begin{aligned} A &= (\beta_1 - 1)K_1Y^{\beta_1} + (\beta_2 - 1)K_2Y^{\beta_2} \\ Y &= A + K_1Y^{\beta_1} + K_2Y^{\beta_2} \\ 0 &= \beta_1(1 - b - \beta_2)K_1Y^{\beta_1} + \beta_2(1 - b - \beta_1)K_2Y^{\beta_2}. \end{aligned}$$

Eliminating K_1 and K_2, it is easy to verify that the solution is such that $\lim_{\alpha \to 1} Y = A_0$.

Case 2: $r < \theta$. On some interval $(1, 1 + \varepsilon)$, with $\varepsilon > 0$ independent of α, Γ' is negative, so is Φ_1', which implies that Φ_1 is negative on $(1, y)$ for some $y > 1$ independent of α. Since $\lim_{\infty} \Phi_1 = \infty$, function Φ_1 must have a root that is greater than y. We conclude that $\lim_{\alpha \to 1} \varpi > y > 1$. Next, we show that $\lim_{\alpha \to 1} Y > A_0 (< A_0)$, whenever $b > 1 (b < 1)$. For $x \geq 1$, define

$$\begin{aligned} P(x) &= \beta_2(\beta_1 - 1)(1 - b - \beta_1)x^{\beta_1} - \beta_1(\beta_2 - 1)(1 - b - \beta_2)x^{\beta_2} \\ Q(x) &= \beta_1\beta_2((1 - b - \beta_1)x^{\beta_1 - 1} - (1 - b - \beta_2)x^{\beta_2 - 1}). \end{aligned}$$

Note that $P(1) = (\beta_1 - \beta_2)(1 - b - \beta_1\beta_2)$, $Q(1) - P(1) = (b - 1)(\beta_1 - \beta_2)$ and for $x > 1$, $P(x) > 0$ and $Q(x) > 1$. From the definition of Φ_1 in Appendix D1. we have

$$\Phi_1(x) = P(x) - P(1) - (Q(x) - Q(1)).$$

Since $\Phi_1(\varpi) = 0$, we have $P(\varpi) - P(1) = Q(\varpi) - Q(1)$, and

$$Y = \frac{X}{\varpi} = \frac{AQ(\varpi)}{P(\varpi)} = A(1 + \frac{(b-1)(\beta_1 - \beta_2)}{P(\varpi)}).$$

Recall that $A_0 = A(1 + \frac{b-1}{1-b-\beta_1\beta_2})$, so in order to show the result, it is enough to show that

$$\frac{\beta_1 - \beta_2}{P(\varpi)} > \frac{1}{1 - b - \beta_1\beta_2} \Leftrightarrow P(1) > P(\varpi).$$

From Appendix D1. we know that on $(1, \varpi)$, $\Phi_1(x) < 0$, and for $x > x^*$, $\Phi_1'(x) > 0$, where $1 < x^* < \varpi$ is defined in Appendix D1. It is also easy to check that for $x > x^*$, $P'(x) > 0$ and $Q'(x) > 0$. We now show **by contradiction** that for all $x \in (x^*, \varpi)$, $Q(x) - Q(1) < 0$. Assume it is not the case. Since Q is decreasing on $[1, x^*]$, there exists y in (x^*, ϖ), such that $Q(y) = Q(1)$. We want to show that

$$P(y) - P(1) > 0,$$

which contradicts the fact that $\Phi(y) < 0$. We have $P(y) > P(1)$ if and only if

$$\beta_2(\beta_1 - 1)(1 - b - \beta_1)\left(y^{\beta_1} - 1\right) > \beta_1(\beta_2 - 1)(1 - b - \beta_2)(y^{\beta_2} - 1)$$

Since y satisfies $(1 - b - \beta_1)(y^{\beta_1 - 1} - 1) = (1 - b - \beta_2)(y^{\beta_2 - 1} - 1)$, it is equivalent to show that

$$-\beta_2(\beta_1 - 1)\frac{y^{\beta_1} - 1}{y^{\beta_1 - 1} - 1} > -\beta_1(\beta_2 - 1)\frac{1 - y^{\beta_2}}{1 - y^{\beta_2 - 1}},$$

and it is enough to show that for $x > 1$, the function Υ is positive where

$$\begin{aligned} \Upsilon(x) &= -\beta_2(\beta_1 - 1)x^{\beta_1} + \beta_1(\beta_2 - 1)x^{\beta_2} + (\beta_1 - \beta_2)x^{\beta_1 + \beta_2 - 1} \\ &\quad \beta_1(\beta_2 - 1)x^{\beta_1 - 1} - \beta_2(\beta_1 - 1)x^{\beta_2 - 1} + \beta_1 - \beta_2. \end{aligned}$$

$$\begin{aligned} \Upsilon'(x) &= -\beta_2\beta_1(\beta_1 - 1)x^{\beta_1 - 1} + \beta_2\beta_1(\beta_2 - 1)x^{\beta_2 - 1} + (\beta_1 - \beta_2)(\beta_1 + \beta_2 - 1)x^{\beta_1 + \beta_2 - 2} \\ &\quad \beta_1(\beta_2 - 1)(\beta_1 - 1)x^{\beta_1 - 2} - \beta_2(\beta_1 - 1)(\beta_2 - 1)x^{\beta_2 - 2}. \end{aligned}$$

$\Upsilon'(1) = 0$ and $\Upsilon'(x)$ has the same sign as $\Sigma(x)$ where

$$\begin{aligned} \Sigma(x) &= -\beta_2\beta_1(\beta_1 - 1)x^{1 - \beta_2} + \beta_2\beta_1(\beta_2 - 1)x^{1 - \beta_1} + (\beta_1 - \beta_2)(\beta_1 + \beta_2 - 1) \\ &\quad \beta_1(\beta_2 - 1)(\beta_1 - 1)x^{-\beta_2} - \beta_2(\beta_1 - 1)(\beta_2 - 1)x^{-\beta_1}. \end{aligned}$$

Then for $x > 1$, $\Sigma'(x) = -\beta_2\beta_1(\beta_1 - 1)(1 - \beta_2)(x - 1)x^{-(\beta_1 + 1)}(x^{\beta_1 - \beta_2} - 1) > 0$. The function Σ is strictly increasing with $\Sigma(1) = 0$ so for $x > 1$, $\Sigma(x) > 0$ and $\Upsilon'(x) > 0$. Since $\Upsilon(1) = 0$, we find that $\Upsilon(x) > 0$ for $x > 1$ and the desired contradiction follows. ■

System S'. In Appendix D2., for $0 < \alpha < 1$, we establish the existence and uniqueness the root $\varpi > 1$ of the equation $\Phi_\alpha(x) = 0$, where Φ_α is defined in Appendix D2. When $\alpha = 1$, the previous equation has at most two solutions, including value 1. We want to show that actually $x = 1$ is the unique solution of $\Phi_1(x) = 0$, so that $\lim_{\alpha \to 1} \varpi = 1$ and $\lim_{\alpha \to 1} X = \lim_{\alpha \to 1} Y$. In Appendix D2., we have seen that $\Phi_1(1) = 0$ and Φ_1' has the same sign as $\Phi_1'(x)$ has the same sign as $\Theta(x)$ and in the case where $\alpha = 1$ we have

$$\Theta'(x) = \beta_1\beta_2(\beta_1 - 1)(\beta_2 - 1)(x - 1)x^{-\beta_1 - 1}(x^{\beta_1 - \beta_2} - 1) > 0 \text{ if } x > 1.$$

Θ is increasing with $\Theta(1) = 0$, so for $x > 1$, we have $\Theta(x) > 0$, so $\Phi_1'(x) > 0$ and thus finally $\Phi_1(x) > 0$. This shows that $x = 1$ is the only root. Let us now determine the limit of Y when α goes to 1. For α given, let us write $(K_1(\alpha), K_2(\alpha), X(\alpha), Y(\alpha))$ the solution of system S'. If $\alpha = 1$, $X(1) = Y(1)$ and system S' becomes a 2 by 2 system

$$\begin{aligned} A &= (\beta_1 - 1)K_1(1)Y(1)^{\beta_1} + (\beta_2 - 1)K_2(1)Y(1)^{\beta_2} \\ Y(1) &= A + K_1(1)Y(1)^{\beta_1} + K_2(1)Y(1)^{\beta_2}. \end{aligned}$$

Next, we assume the following asymptotic expansion in α for the solution around $\alpha = 1$

$$\begin{aligned} K_1(\alpha) &= K_1(1)(1 + k_1 h + o(h)) \\ K_2(\alpha) &= K_2(1)(1 + k_2 h + o(h)) \\ X(\alpha) &= Y(1)(1 + xh + o(h)) \\ Y(\alpha) &= Y(1)(1 + yh + o(h)), \end{aligned}$$

where (k_1, k_2, x, y) are numbers independent of α, $h = 1 - \alpha$. Plugging back these asymptotic expressions into system S' and taking a first order Taylor expansion yields

$$\begin{aligned} (x-1)Y(1) &= (k_1 + \beta_1 x)K_1(1)Y(1)^{\beta_1} + (k_2 + \beta_2 x)K_2(1)Y(1)^{\beta_2} \\ 0 &= (\beta_1 - 1)(k_1 + \beta_1 x)K_1(1)Y(1)^{\beta_1} + (\beta_2 - 1)(k_2 + \beta_2 x)K_2(1)Y(1)^{\beta_2} \\ yY(1) &= (k_1 + \beta_1 y)K_1(1)Y(1)^{\beta_1} + (k_2 + \beta_2 y)K_2(1)Y(1)^{\beta_2} \\ 0 &= (\beta_1 - 1)(k_1 + \beta_1 y)K_1(1)Y(1)^{\beta_1} + (\beta_2 - 1)(k_2 + \beta_2 y)K_2(1)Y(1)^{\beta_2}. \end{aligned}$$

Manipulating these relationships we find that

$$\begin{aligned} (\beta_1 - \beta_2)K_1(1)Y(1)^{\beta_1} &= -(\beta_2 - 1)Y(1) + \beta_2 A \\ (\beta_1 - \beta_2)K_2(1)Y(1)^{\beta_2} &= (\beta_1 - 1)Y(1) - \beta_1 A \\ \beta_1(\beta_1 - \beta_2)(y - x)K_1(1)Y(1)^{\beta_1} &= (\beta_2 - 1)(1 + y - x)Y(1) \\ \beta_2(\beta_1 - \beta_2)(y - x)K_2(1)Y(1)^{\beta_2} &= -(\beta_1 - 1)(1 + y - x)Y(1). \end{aligned}$$

Eliminating $K_1(1)Y(1)^{\beta_1}, K_2(1)Y(1)^{\beta_2}, y - x$ leads to

$$\frac{-\beta_1(\beta_2 A - (\beta_2 - 1)Y(1))}{\beta_2((\beta_1 - 1)Y(1) - \beta_1 A)} = \frac{\beta_2 - 1}{-(\beta_1 - 1)}.$$

It follows easily that

$$Y(1) = \frac{\beta_1 \beta_2 A}{(\beta_1 - 1)(\beta_2 - 1)} = \frac{1}{r}. \blacksquare$$

Step 3: Proof of Proposition 1. Case $\theta \le r$. Given step 2, for the solution of system S, we know that $\lim_{\alpha \to 1} X = \lim_{\alpha \to 1} Y = A_0$. When $F_2(M, M) = 0$ is optimal, we show in Appendix F that solution Y satisfies

$$\frac{\partial Y}{\partial \alpha} < 0 \ (\frac{\partial Y}{\partial \alpha} > 0) \text{ whenever } b < 1 \ (b > 1),$$

and when $b = 1$, $Y = A_0$. Since $\lim_{\alpha \to 1} Y = A_0$, this implies that for all $\alpha \le 1$, when $b > 1$ ($b < 1$) we have $Y \le A_0$ ($Y \ge A_0$). Condition (35) is satisfied, and the desired result follows. $\blacksquare$

Case $\theta > r$. Given step 2, for the solution of system S, we know that $\lim_{\alpha \to 1} Y > A_0 (< A_0)$, whenever

$b > 1(b < 1)$, and the solution of system S' satisfies $\lim_{\alpha\to 1} Y = \frac{1}{r} > A_0$. From step 1, for small values of α, $F_2(M,M) = 0$ is optimal. When $b > 1(b < 1)$, as long as $F_2(M,M) = 0$ is optimal, we know that $\frac{\partial Y}{\partial \alpha} > 0(< 0)$. Since for $b > 1(b < 1)$ the solution Y of system S is such that $\lim_{\alpha\to 1} Y > A_0(< A_0)$, around $\alpha = 1$, condition $Y \le A_0(\ge A_0)$ is violated. By continuity of the solution in α, there must exist a critical value $\alpha^* \in (0,1]$ such that for all $\alpha < \alpha^*$, for $b > 1(b < 1)$ we have $Y < A_0(> A_0)$ and $F_2(M,M) = 0$ is optimal. At $\alpha = \alpha^*$, $Y = A_0$, we have $F_2(M,M) = 0$ *and* $z^*(1) = 0$. Then, for $\alpha > \alpha^*$, we must have $z^*(1) = 0$. In Appendix F, when $z^*(1) = 0$ is imposed, we show that $\frac{\partial Y}{\partial \alpha} > 0$. Hence, for all $\alpha \in (\alpha^*, 1)$, $Y < \frac{1}{r}$ and therefore at $W = M$, we have $dW_t = \left(r - \frac{1}{Y}\right) W_t dt < 0$: **Wealth can never achieve a new maximum.** ■

APPENDIX F

Proof of Proposition 2

Case 1: $F_2(M,M) = 0$. When α increases $f(1)$ must decrease. Since $f'(1) = (1-b)f(1)$, we deduce that when $b < 1(b > 1)$, $\frac{\partial Y}{\partial \alpha} < 0$ ($\frac{\partial Y}{\partial \alpha} > 0$). Hence $\frac{1}{Y-A}\frac{\partial Y}{\partial \alpha} > 0, b \neq 1$. When $b = 1$, then $Y = A$, so $\frac{\partial Y}{\partial \alpha} = 0$. Then

$$\begin{aligned}
\frac{\partial K_1}{\partial \alpha} Y^{\beta_1+1} &= (-\beta_1 K_1 Y^{\beta_1} + \frac{\beta_2(1-b-\beta_1)Y}{(\beta_1-\beta_2)(b-1)})\frac{\partial Y}{\partial \alpha} \\
&= \frac{K_1 Y^{\beta_1}}{Y-A}\frac{\partial Y}{\partial \alpha}(\beta_1 A + (1-\beta_1)Y).
\end{aligned}$$

Recall that $z^*(1) \ge 0$ so $A \ge Y + (\beta_1 - 1)K_1 Y^{\beta_1} + (\beta_2 - 1)K_2 Y^{\beta_2}$. Since $Y = A + K_1 Y^{\beta_1} + K_2 Y^{\beta_2}$, it follows that $\beta_1 A + (1-\beta_1)Y \ge (\beta_2 - \beta_1)K_2 Y^{\beta_2} > 0$. Therefore $\frac{\partial K_1}{\partial \alpha} > 0$. Differentiating relationship (12) with respect to α leads to

$$\begin{aligned}
&\left(f'(u)\right)^{-1}\left(\frac{A}{b}\left(f'(u)\right)^{-\frac{1}{b}} - \frac{\beta_1-1}{b}K_1\left(f'(u)\right)^{\frac{\beta_1-1}{b}} - \frac{\beta_2-1}{b}K_2\left(f'(u)\right)^{\frac{\beta_2-1}{b}}\right)\frac{\partial f'(u)}{\partial \alpha} \qquad (36) \\
= \;& \frac{\partial K_1}{\partial \alpha}\left(f'(u)\right)^{\frac{\beta_1-1}{b}} + \frac{\partial K_2}{\partial \alpha}\left(f'(u)\right)^{\frac{\beta_2-1}{b}}.
\end{aligned}$$

The sign of the LHS is the same as $\frac{\partial K_1}{\partial \alpha}\left(f'(u)\right)^{\frac{\beta_1-\beta_2}{b}} + \frac{\partial K_2}{\partial \alpha}$ and $u \mapsto \frac{\partial K_1}{\partial \alpha}\left(f'(u)\right)^{\frac{\beta_1-\beta_2}{b}} + \frac{\partial K_2}{\partial \alpha}$ achieves its minimum at $u = 1$ and its maximum at $u = \alpha$. Then

$$\frac{\partial K_1}{\partial \alpha}Y^{\beta_1-\beta_2} + \frac{\partial K_2}{\partial \alpha} = Y^{-(1+\beta_2)}\frac{\partial Y}{\partial \alpha}(A - (\beta_1-1)K_1Y^{\beta_1} - (\beta_2-1)K_2Y^{\beta_2}).$$

Since $A - (\beta_1-1)K_1Y^{\beta_1} - (\beta_2-1)K_2Y^{\beta_2} > 0$, when $b > 1$, $\frac{\partial Y}{\partial \alpha} > 0$, so $\frac{\partial K_1}{\partial \alpha}Y^{\beta_1-\beta_2} + \frac{\partial K_2}{\partial \alpha} > 0$. In addition, when $b = 1$, $\frac{\partial K_1}{\partial \alpha}Y^{\beta_1-\beta_2} + \frac{\partial K_2}{\partial \alpha} = 0$. Hence, if $b \ge 1$, $\frac{\partial f'(u)}{\partial \alpha} > 0$. As $c^* = M\left(f'(u)\right)^{-\frac{1}{b}}$, $\frac{\partial c^*}{\partial \alpha} < 0$. Conversely, when $b < 1$, we have $\frac{\partial K_1}{\partial \alpha}Y^{\beta_1-\beta_2} + \frac{\partial K_2}{\partial \alpha} < 0$. Then

$$\begin{aligned}
\frac{\partial K_1}{\partial \alpha}X^{\beta_1-\beta_2} + \frac{\partial K_2}{\partial \alpha} &= \frac{X^{-\beta_2}}{(Y-A)Y}\frac{\partial Y}{\partial \alpha}\Big(K_1X^{\beta_1}(Y(1-\beta_1)+\beta_1A) + K_2X^{\beta_2}(Y(1-\beta_2)+\beta_2A)\Big) \\
&= \frac{X^{-\beta_2}}{(Y-A)Y}\frac{\partial Y}{\partial \alpha}A(\alpha X - Y) > 0.
\end{aligned}$$

Hence, there exists u_α^* in $(\alpha,1)$, so that $\frac{\partial f'(u)}{\partial \alpha} > 0$ on $[\alpha, u_\alpha^*)$ and $\frac{\partial f'(u)}{\partial \alpha} < 0$ on $(u_\alpha^*, 1]$. We conclude that $\frac{\partial c^*}{\partial \alpha} < 0$ on $[\alpha, u_\alpha^*)$ and $\frac{\partial c^*}{\partial \alpha} > 0$ on $(u_\alpha^*, 1]$. ■

Case 2: $z^*(1) = 0$. In this case, we have

$$Y\frac{\partial Y}{\partial \alpha} = (\beta_1 K_1 Y^{\beta_1} + \beta_2 K_2 Y^{\beta_2})\frac{\partial Y}{\partial \alpha} + \frac{\partial K_1}{\partial \alpha}Y^{\beta_1+1} + \frac{\partial K_2}{\partial \alpha}Y^{\beta_2+1}.$$

Recall that $Y = \beta_1 K_1 Y^{\beta_1} + \beta_2 K_2 Y^{\beta_2}$, so $\frac{\partial K_1}{\partial \alpha}Y^{\beta_1} + \frac{\partial K_2}{\partial \alpha}Y^{\beta_2} = 0$, and therefore, $\frac{\partial K_1}{\partial \alpha}$ and $\frac{\partial K_2}{\partial \alpha}$ must have opposite signs. Similarly

$$X + \alpha\frac{\partial X}{\partial \alpha} = \frac{\partial K_1}{\partial \alpha}X^{\beta_1} + \frac{\partial K_2}{\partial \alpha}X^{\beta_2} + \frac{1}{X}(\beta_1 K_1 X^{\beta_1} + \beta_2 K_2 X^{\beta_2})\frac{\partial X}{\partial \alpha}.$$

Since $\alpha X = \beta_1 K_1 X^{\beta_1} + \beta_2 K_2 X^{\beta_2}$, we find that $\frac{\partial K_1}{\partial \alpha}X^{\beta_1} + \frac{\partial K_2}{\partial \alpha}X^{\beta_2} = X > 0$. It remains to investigate the sign of $\frac{\partial K_1}{\partial \alpha}(f'(u))^{\frac{\beta_1-\beta_2}{b}} + \frac{\partial K_2}{\partial \alpha}$. Set $y = (f'(u))^{\frac{1}{b}}$ and for y in $[Y, X]$ define the auxiliary function

$$\Delta: \quad y \quad \mapsto \quad \frac{\partial K_1}{\partial \alpha}y^{\beta_1-\beta_2} + \frac{\partial K_2}{\partial \alpha},$$

Δ is a continuous monotonic function with $\Delta(Y) = 0$ and $\Delta(X) > 0$. Hence, we must have $\frac{\partial K_1}{\partial \alpha} > 0$ and $\frac{\partial K_2}{\partial \alpha} < 0$ and Δ is positive on $[Y, X]$. We conclude that $\frac{\partial c^*}{\partial \alpha} < 0$, and in particular $\frac{\partial Y}{\partial \alpha} > 0$. ■

APPENDIX G

Proof of Proposition 3. Note that

$$\frac{\partial}{\partial y}\left(\frac{z^*}{W}\right) = -\frac{(\mu - r)y^{\beta_1-1}}{b\sigma^2}\left(\frac{A(\beta_1^2 K_1 + \beta_2^2 K_2 y^{\beta_2-\beta_1}) + K_1 K_2(\beta_1 - \beta_2)^2 y^{\beta_2}}{(A + K_1 y^{\beta_1} + K_2 y^{\beta_2})^2}\right).$$

For y in $[Y, X]$, define the auxiliary function

$$\Psi: \quad y \quad \mapsto \quad A(\beta_1^2 K_1 + \beta_2^2 K_2 y^{\beta_2-\beta_1}) + K_1 K_2(\beta_1 - \beta_2)^2 y^{\beta_2}.$$

Ψ is strictly increasing so it has at most one root. At $u = 1$, we have $\frac{z^*}{W} \geq 0$, so either Ψ has no root and is strictly positive or, Ψ has one root so it is first negative and then positive. We examine the sign of $\Psi(Y)$.

Case 1. When $z^*(1) = 0$, since $z^*(\alpha) = 0$, Ψ indeed must have a root. $\frac{z^*}{W}$ is hump-shaped in u. ■

Case 2. When $F_2(M, M) = 0$, we have

$$\begin{aligned}\Psi(Y) &= A(\beta_1^2 K_1 + \beta_2^2 K_2 Y^{\beta_2-\beta_1}) + K_1 K_2(\beta_1 - \beta_2)^2 Y^{\beta_2} \\ &= \frac{\beta_1(\beta_1 - \beta_2)K_1}{\beta_1 + b - 1}\left(A(\beta_1 + \beta_2 + b - 1) + (1 - b - \beta_2)(1 - b - \beta_1)\frac{Y - A}{b - 1}\right) \\ &= \frac{\beta_1(\beta_1 - \beta_2)K_1}{(1 - b - \beta_2)(b - 1)}\left(\frac{1}{\theta} - Y\right),\end{aligned}$$

as $-\frac{\beta_1\beta_2 A}{(\beta_1+b-1)(1-b-\beta_2)} = \frac{1}{\theta}$. Hence $\Psi(Y) > 0$ iff $Y < \frac{1}{\theta}$ ($Y > \frac{1}{\theta}$) whenever $b > 1$ ($b < 1$). ■

Case $b = 1$. We have $Y = A$ and

$$\begin{aligned}\Psi(Y) &= K_1(\beta_1 - \beta_2)(A(\beta_1 + \beta_2 - (\beta_1 - \beta_2)K_1 Y^{\beta_1}) \\ &= \frac{AK_1(\beta_1 - \beta_2)}{\varpi^{\beta_1}}\Big((\beta_1 + \beta_2)\varpi^{\beta_1} - (\beta_2 - \alpha(\beta_2 - 1)\varpi)\Big),\end{aligned}$$

where ϖ is defined by relationship (28). $\frac{z^*}{W}$ is strictly increasing (hump shaped) in u exactly when $\Psi(Y) \geq 0 (\leq 0)$. ■

APPENDIX H

Proof of Proposition 4. Let $y = f'(u)^{\frac{1}{b}}$. From relationship (36), we have

$$\frac{1}{y}\frac{\partial y}{\partial \alpha} = \frac{y^{\beta_1}\frac{\partial K_1}{\partial \alpha} + y^{\beta_2}\frac{\partial K_2}{\partial \alpha}}{A - (\beta_1 - 1)K_1 y^{\beta_1} - (\beta_2 - 1)K_2 y^{\beta_2}}.$$

Since $z^* = \frac{\mu - r}{b\sigma^2}\left(Ay^{-1} - (\beta_1 - 1)K_1 y^{\beta_1 - 1} - (\beta_2 - 1)K_2 y^{\beta_2 - 1}\right)$, it follows that

$$\begin{aligned}\frac{\partial z^*}{\partial \alpha} &= -\frac{\mu - r}{b\sigma^2 y}\left((\beta_1 - 1)\frac{\partial K_1}{\partial \alpha}y^{\beta_1} + (\beta_2 - 1)\frac{\partial K_2}{\partial \alpha}y^{\beta_2} + (A + (\beta_1 - 1)^2 K_1 y^{\beta_1} + (\beta_2 - 1)^2 K_2 y^{\beta_2}\frac{\partial y}{\partial \alpha}\right) \\ &= -\frac{(\mu - r)y^{\beta_1 + \beta_2}}{b\sigma^2 y}\frac{A(\beta_1\frac{\partial K_1}{\partial \alpha}y^{-\beta_2} + \beta_2\frac{\partial K_2}{\partial \alpha}y^{-\beta_1}) + (\beta_1 - \beta_2)((\beta_1 - 1)K_1\frac{\partial K_1}{\partial \alpha} - (\beta_2 - 1)K_2\frac{\partial K_2}{\partial \alpha}}{A - (\beta_1 - 1)K_1 y^{\beta_1} - (\beta_2 - 1)K_2 y^{\beta_2}}.\end{aligned}$$

The denominator of the above fraction is positive. To investigate the sign of its numerator, let us for y in $[Y, X]$, define the auxiliary function

$$\Theta : \quad y \quad \mapsto \quad A(\beta_1\frac{\partial K_1}{\partial \alpha}y^{-\beta_2} + \beta_2\frac{\partial K_2}{\partial \alpha}y^{-\beta_1}) + (\beta_1 - \beta_2)((\beta_1 - 1)K_1\frac{\partial K_1}{\partial \alpha} - (\beta_2 - 1)K_2\frac{\partial K_2}{\partial \alpha}).$$

Θ is differentiable and $\Theta'(y) = -\beta_1\beta_2 A(\frac{\partial K_1}{\partial \alpha}y^{-\beta_2 - 1} + \frac{\partial K_2}{\partial \alpha}y^{-\beta_1 - 1})$. Since $\frac{\partial K_1}{\partial \alpha} > 0$ and $\frac{\partial K_2}{\partial \alpha} < 0$, Θ' is strictly increasing and either $\Theta(Y) \geq 0$ or Θ achieves its minimum at y^* such that $\frac{\partial K_1}{\partial \alpha}(y^*)^{\beta_1} + \frac{\partial K_2}{\partial \alpha}(y^*)^{\beta_2} = 0$. If Θ achieves its minimum at y^*, then $\Theta(y^*) = (\beta_1 - \beta_2)(y^*)^{-\beta_2}\frac{\partial K_1}{\partial \alpha} > 0$, and in this case, $\Theta > 0$ on $[Y, X]$. Otherwise, $\Theta' > 0$ and Θ is strictly increasing. To prove that $\Theta \geq 0$ on $[Y, X]$, it is enough to show that $\Theta(Y) \geq 0$ or equivalently that $z^*(1)$ is a decreasing function of α.

Case 1: $F_2(M, M) = 0$. In this case, $z^*(1) = \frac{\mu - r}{b\sigma^2 Y}\left(A - (\beta_1 - 1)K_1 Y^{\beta_1} - (\beta_2 - 1)K_2 Y^{\beta_2}\right)$. If $b = 1$, then $Y = A$ and

$$\frac{\partial z^*(1)}{\partial \alpha} = -\frac{\mu - r}{b\sigma^2 A}\left((\beta_1 - 1)\frac{\partial K_1}{\partial \alpha}A^{\beta_1} + (\beta_2 - 1)\frac{\partial K_2}{\partial \alpha}A^{\beta_2}\right) < 0.$$

If $b \neq 1$, we have $z^*(1) = \frac{\mu - r}{b\sigma^2}\left(1 + \frac{\beta_1\beta_2(Y - A)}{(b - 1)Y}\right)$. Therefore

$$\frac{\partial z^*(1)}{\partial \alpha} = \frac{\mu - r}{b\sigma^2}\left(-\frac{\beta_1\beta_2 A}{Y^2}\frac{\frac{\partial Y}{\partial \alpha}}{1 - b}\right) < 0. \blacksquare$$

Case 2: $z^*(1) = 0$. We have $\frac{\partial z^*(1)}{\partial \alpha} = 0$ and $\Theta(Y) = 0$. The proof is complete. ■

APPENDIX I

Representation of $\frac{c^*}{M}, c^*$ and W as Stochastic Processes

Process $\frac{c^*}{M}$. For u in $(\alpha, 1)$, recall that $u = G(\frac{c^*}{M})$ so denoting $H = G^{-1}$, we have $H'(u) = \frac{1}{G'(\frac{c^*}{M})}$ and $H''(u) = -\frac{G''(\frac{c^*}{M})}{(G'(\frac{c^*}{M}))^3}$. Applying Ito lemma, we find

$$\begin{aligned} d\left(\frac{c_t^*}{M_t}\right) &= H'(u_t)du_t + \frac{\sigma^2}{2}\left(\frac{z_t}{M_t}\right)^2 H''(u_t)dt \\ &= \frac{rG(\frac{c_t^*}{M_t}) - \frac{c_t^*}{M_t} + \frac{(\mu - r)^2}{b\sigma^2}c_t^* G'(\frac{c_t^*}{M_t}) - \frac{1}{2}\left(\frac{\mu - r}{b\sigma}\right)^2\left(\frac{c_t^*}{M_t}\right)^2 G''(\frac{c_t^*}{M_t})}{G'(\frac{c_t^*}{M_t})}dt \\ &\quad + \frac{\mu - r}{b\sigma}\frac{c_t^*}{M_t}dw_t. \end{aligned}$$

Then

$$\begin{aligned}
& rG(\frac{c_t^*}{M_t}) - \frac{c_t^*}{M_t} - \frac{1}{2}\left(\frac{\mu-r}{b\sigma}\right)^2 \left(\frac{c_t^*}{M_t}\right)^2 G''(\frac{c_t^*}{M_t}) \\
= \ & (r-\frac{1}{A})\left((r-\frac{\beta_1(\beta_1-1)}{2}\left(\frac{\mu-r}{b\sigma}\right)^2)K_1\left(\frac{c_t^*}{M_t}\right)^{1-\beta_1}\right. \\
& \left. +(r-\frac{\beta_2(\beta_2-1)}{2}\left(\frac{\mu-r}{b\sigma}\right)^2)K_2\left(\frac{c_t^*}{M_t}\right)^{1-\beta_2} + A\frac{c_t^*}{M_t}\right) \\
= \ & (r-\frac{1}{A})\frac{c_t^*}{M_t}G'(\frac{c_t^*}{M_t}).
\end{aligned}$$

So

$$d\left(\frac{c_t^*}{M_t}\right) = \frac{c_t^*}{M_t}\left((r-\frac{1}{A}+\frac{(\mu-r)^2}{b\sigma^2})dt + \frac{\mu-r}{b\sigma}dw_t\right).$$

For u in $[0,1]$, we have $\frac{1}{X} \leq \frac{c^*}{M} \leq \frac{1}{Y}$. Define the geometric Brownian motion v such that $v_0 = X(f'(\frac{W_0}{M_0}))^{-\frac{1}{b}}$ and

$$dv_s = v_s\left((r-\frac{1}{A}+\frac{(\mu-r)^2}{b\sigma^2})ds + \frac{\mu-r}{b\sigma}dw_s\right).$$

A representation of the process $\frac{c^*}{M}$ (see Harrison [23] p 22) is

$$\frac{c_t^*}{M_t} = \frac{v_t e^{L_t-U_t}}{X},$$

where the processes L and U are increasing and continuous with $L_0 = U_0 = 0$ and

$$\begin{aligned}
L_t &= \sup_{0\leq s\leq t}[\log v_s - U_s]^- \\
U_t &= \sup_{0\leq s\leq t}\left[\log\frac{X}{Y} - \log v_s - L_s\right]^- . \ \blacksquare
\end{aligned}$$

Consumption and Wealth Processes.

Case 1: $F_2(M,M) = 0$. Using Ito lemma for semimartingales, we find that

$$\log H\left(\frac{W_t}{M_t}\right) = \log H\left(\frac{W_0}{M_0}\right) + (\frac{r-\theta}{b}+\frac{(\mu-r)^2}{2b\sigma^2})t + \frac{\mu-r}{b\sigma}w_t - \frac{H'(1)}{H(1)}\log\frac{M_t}{M_0},$$

Note that $\frac{H'(1)}{H(1)} = \frac{Y}{G'(\frac{1}{Y})} > 0$. As explained in Grossman and Zhou [21], the quantity $\log H\left(\frac{W_t}{M_t}\right)$ is bounded from above by $\log H(1) = -\log Y$ and $\frac{H'(1)}{H(1)}\log\left(\frac{M_t}{M_0}\right) \geq 0$ serves as a regulator to keep the arithmetic Brownian motion from exceeding $-\log Y$. Define

$$l_t = \sup_{0\leq s\leq t}\left[\log H\left(\frac{W_0}{M_0}\right) + (\frac{r-\theta}{b}+\frac{(\mu-r)^2}{2b\sigma^2})s + \frac{\mu-r}{b\sigma}w_s + \log Y\right]^+.$$

It follows that $M_t = M_0 e^{\frac{Y}{G'(\frac{1}{Y})}l_t}$. Recall that $W_t = M_t G(\frac{c_t^*}{M_t})$ and $c_t^* = M_t H(\frac{W_t}{M_t})$, so a representation for the wealth and consumption processes is

$$\begin{aligned}
W_t &= M_0 e^{\frac{Y}{G'(\frac{1}{Y})}l_t} G\left(H\left(\frac{W_0}{M_0}\right)e^{(\frac{r-\theta}{b}+\frac{(\mu-r)^2}{2b\sigma^2})t+\frac{\mu-r}{b\sigma}w_t-l_t}\right) \\
c_t^* &= H\left(\frac{W_0}{M_0}\right)e^{(\frac{r-\theta}{b}+\frac{(\mu-r)^2}{2b\sigma^2})t+\frac{\mu-r}{b\sigma}w_t-l_t}.
\end{aligned}$$

Case 2: $z^*(1) = 0$. The wealth process can never exceed its all time high M_0. Consumption and wealth processes are given by

$$\begin{aligned} c_t^* &= \frac{M_0}{X} v_t e^{L_t - U_t} \\ W_t &= M_0 G\left(\frac{v_t e^{L_t - U_t}}{X}\right). \blacksquare \end{aligned}$$

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Tables

Table I: Disentangling hedging and ratchet effects

α	u	0.6	0.65	0.7	0.75	0.8	0.85	0.9	0.925	0.95	0.975	1
0.6	$\frac{\widehat{z}^*}{W}$	0	0.283	0.383	0.450	0.499	0.537	0.568	0.581	0.592	0.603	0.613
	$\frac{z^*}{W}$	0	0.279	0.375	0.436	0.478	0.506	0.526	0.532	0.537	0.540	0.541
0.8	$\frac{\widehat{z}^*}{W}$	-	-	-	-	0	0.248	0.339	0.372	0.402	0.428	0.450
	$\frac{z^*}{W}$	-	-	-	-	0	0.230	0.299	0.318	0.331	0.336	0.334
0.9	$\frac{\widehat{z}^*}{W}$	-	-	-	-	-	-	0	0.168	0.234	0.283	0.322
	$\frac{z^*}{W}$	-	-	-	-	-	-	0	0.143	0.184	0.199	0.190

Footnotes

1. In the US, trustees and charity professionals who run foundations are only obliged to spend as little as 5% a year of the capital. In many foundations, capricious and poorly thought out projects or programs were undertaken to fulfill the interests of trustee managers not the wishes of the founder [17].

2. The high-water mark is a target value that can depend on the current asset value of the fund. It is periodically adjusted due to withdraws, allocated expenses and a contractual growth rate. In the simplest case, the high-water mark is the highest level the asset has reached in the past.

3. This constraint was first introduced by Grossman and Zhou [21] to examine the problem of maximizing the long term growth rate of expected utility of final wealth. Their analysis is quite insightful but they do not allow for endogenous withdraws from the fund to finance intermediate consumption. Cvitanic and Karatzas [12] extend their work to a more general class of stochastic processes.

4. See for instance such Sundaresan [32], Constantidines [8], Detemple and Zapatero [13], Campbell and Cochrane [5] .

5. For instance see Baski and Chen [1] and Smith [31].

6. Cvitanic and Karatzas [11] and Cuoco [9] develop a general martingale approach to cope with convex contemporaneous constraints on trading strategies which includes the case of incomplete markets and prohibited short sales. Cuoco and Liu [10] analyze the optimal consumption portfolio choice problem under margin requirements and evaluate the cost of the constraint. He and Pages [24] and El Karaoui and JeanBlanc-Picqué [18] treat the case of non-negative wealth in presence of labor income. Grossman and Villa [20] followed by Villa and Zariphopoulou [35] study the consumption-portfolio problem for a CRRA investor facing a leverage constraint.

7. This assumption is only needed for the logarithmic preferences case. When $u(c)$ has a constant sign, Lebesgue Monotone Convergence Theorem can be applied directly to prove step 4 of the Verification Theorem provided in section 4 of the paper.

8. The strict concavity of F comes from the fact that the utility function is strictly concave and the constraint is linear so that if W and $W'$are admissible wealth processes, then for all λ in $[0,1]$, $\lambda W + (1-\lambda)W'$ is also admissible.

9. In the Appendix, we show that the value function F is globally concave iff $(\beta_1 - 1)K_1 Y^{\beta_1} + (\beta_2 - 1)K_2 Y^{\beta_2} \leq A$.

10. This property is actually necessary for a well defined problem. In the sequel, when the drawdown constraint (3) is imposed, restrictions on the parameters of the model are made so that this "reflecting condition" is satisfied.

11. In the limit case $\alpha = 1$, when $F_2(M,M) = 0$ is optimal, we have $X = Y = A_0$. Since for $b < 1$, $\frac{1}{A_0} > \frac{1}{A}$, by continuity in α, for α large enough, it follows that $\alpha X < A$, which implies that $\frac{c^*}{W} > \frac{1}{A}$, for all u in $[\alpha, 1]$.

12. For the definition of a regulated Brownian motion, see Harrison [23], p14.

13. The University of Newcastle upon Tyne study by Parker, Pearce and Tiffin [29] indicates that women are twice as likely to be downwardly mobile. The study involved men and women born in 1947 in Newcastle and followed them from childhood to age 50. Researchers noted the findings might be explained by the fact that men born during that era gained much of their self-esteem from their careers, whereas women found fulfillment from social pursuits outside of work, such as children and friendships.

14. The choice of the functional form of the utility function is motivated by tractability reasons.